Computing Equilibria in Anonymous Games

Constantinos Daskalakis and Christos Papadimitriou

FOCS 2007.

Speaker: Joseph Chuang-Chieh Lin

Institute of Information Science
Academia Sinica
Taiwan

6 January 2017

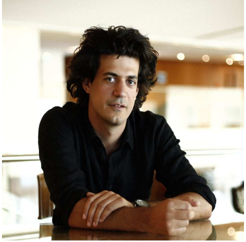
References

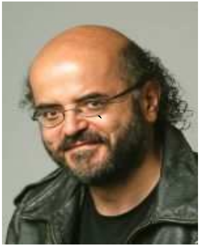
- Constantinos Daskalakis and Christos Papadimitriou: "Computing Equilibria in Anonymous Games." *FOCS* 2007.
- Constantinos Daskalakis and Christos Papadimitriou: "Discretized Multinomial Distributions and Nash Equilibria in Anonymous Games." *FOCS* 2008.

⋆ Yu Cheng, Ilias Diakonikolas, Alistair Stewart: "Playing Anonymous Games using Simple Strategies." *SODA 2017*.

Outline

Authors





"Will you come to FOCS? This decision depends on many factors, but one of them is HOW MANY other theoreticians will come."

PTAS for two-strategy anonymous games

The main idea:

- Round the mixed strategies of the players to some nearby multiple of ϵ.
- Each such quantized mixed strategy can be considered as a pure strategy.
- Exhaustively search for the solution (polynomial time in n).

⋆ The only problem: Why should the expected utilities before and after the quantization be close?

Rough idea of the key probabilistic lemma

- Players strategies: n Bernoulli random variables.
  - with probabilities $p_1, p_2, \ldots, p_n$).
- There exists a way to round the probabilities to multiples of $1/z$, for any z, so that:
  - the distribution of the sum of these n random variables is affected only by an additive $O(1/\sqrt{z})$ in total variational (TV) distance.
  - Such a TV distance is independent of n.

▷ $O(n^{1/\epsilon^2})$ PTAS to find an $O(1/\sqrt{z})$-NE for a two-strategy anonymous game.

The total variation distance (recall)

The total variation distance

$\mathbb{P}, \mathbb{Q}$: two distributions supported by a finite set $\mathcal{A}$.

$$||\mathbb{P} - \mathbb{Q}|| \triangleq ||\mathbb{P} - \mathbb{Q}||_{TV} = \frac{1}{2} \cdot \sum_{\alpha \in \mathcal{A}} |\mathbb{P}(\alpha) - \mathbb{Q}(\alpha)| \,.$$

Recall: For $f : \{0, \ldots, n\} \mapsto [0, 1]$,

$$\sum_{\alpha \in \mathcal{A}} f(\alpha) \cdot (\mathbb{P}(\alpha) - \mathbb{Q}(\alpha)) \leq 2\delta.$$

if $||\mathbb{P} - \mathbb{Q}|| \leq \delta$.

The main theorem

Theorem 1

- $\{p_i\}_{i=1}^n$: arbitrary probabilities, $p_i \in [0,1]$ for $i = 1, \ldots, n$.
- $\{X_i\}_{i=1}^n$: independent indicator random variables, $\mathbf{E}[X_i] = p_i$.
- $z > 0$: a positive integer.

Then there exists another set of probabilities $\{q_i\}_{i=1}^n$, $q_i \in [0,1]$ for $i \in [n]$, which satisfy the following properties:

1. $|q_i - p_i| = O(1/z)$, for all $i \in [n]$.
2. q_i is an integer multiple of $1/z$, for all $i \in [n]$.
3. if $\{Y_i\}_{i=1}^n$ are independent indicator random variables such that $\mathbf{E}[Y_i] = q_i$, then

$$\left\| \sum_i X_i - \sum_i Y_i \right\| = O(z^{-1/2}),$$

and for all $j \in [n]$,

$$\left\| \sum_{i, i \neq j} X_i - \sum_{i, i \neq j} Y_i \right\| = O(z^{-1/2}).$$

The constructive proof for the PTAS for the two-strategy case

Corollary 1

There is a PTAS for finding a mixed NE for the two-player anonymous game.

Sketch of the proof:

- Let $(p_1, \dots, p_n)$ be a mixed NE of the game.
- **Claim**: $q_1, \dots, q_n$ specified by Theorem 1 constitute an $O(1/\sqrt{z})$-approximated mixed NE.
- The absolute change of the expected utility of player i: bounded by $\| \sum_{j \neq i} X_j - \sum_{j \neq i} Y_j \|_{TV}$.
  - The distribution over $\prod_{n-1}^{2}$ defined by $\{p_i\}_{i=1}^{n}$ is replaced by $\{q_i\}_{i=1}^{n}$:
  - ⋆ **Recall**: $\prod_{n-1}^{k} = \{(x_1, \dots, x_k) \in ([k] \cup \{0\})^k \mid \sum_{i=1}^{k} x_i = n-1\}$: the set of all the ways to partition $n-1$ players into the k strategies.
- Yet, how to compute such $\{q_i\}_{i=1}^{n}$?

Sketch of the proof of Corollary 1 (computation of q_i's)

- Remember, q_i is an integer multiple of $1/z$, for each i.
- We proceed with a related $(z+1)$-strategy game, for $z = O(\frac{1}{\epsilon^2})$, and seek for its pure NE.

The new $(z+1)$-strategy game

The j-th pure strategy, $j \in [z] \cup \{0\}$, corresponds to a player in the original game playing strategy 2 w.p. $\frac{j}{z}$.

- The payoffs resulting from the new game: translating the pure strategy profile into a mixed strategy profile of the original game.

Sketch of the proof of Corollary 1 (computation of q_i's)

- For any player i, with its strategy $j \in [z] \cup \{0\}$, and any partition $x \in \prod_{n-1}^{z+1}$, we can compute its payoff by dynamic programming [e.g., Papadimitriou @STOC 2005].
  - $n^{O(z)} = n^{O(1/\epsilon)}$ time overall.
- The remaining details are omitted.

Some naïve methods of rounding seem to fail

- *Rounding to the* **closest multiple** *of* $1/z$.
  - A counterexample: $p_i := 1/n, \forall i$.
  - The trivial rounding make $q_i := 0, \forall i$.
  - ▷ $\| \sum_i X_i - \sum_i Y_i \|_{TV} \to 1 - 1/e$ as $n \to \infty$.
- *Randomized Rounding*:
  - Independently rounding each p_i to some random q_i which is an integer multiple of $1/z$ such that $\mathbf{E}[q_i] = p_i$.
  - Seems promising since $\mathbf{E}[\Pr[\sum_i Y_i = \ell]] = \Pr[\sum_i X_i = \ell]$ (correct expectation).
  - ▷ The trouble: $\mathbf{E}[\Pr[\sum_i Y_i = \ell]]$ is *very small*.
    - Concentration seems to require z scaling polynomially in n.

The intuition of Theorem 1's proof

- The distribution of $\sum_i X_i$ should be close (in TV distance) to a Poisson distribution of the same mean $\sum_i p_i$.
- Hence, if we define q_i's (as multiples of $1/z$) in such a way that the means $\sum_i p_i$ and $\sum_i q_i$ are close, then the distribution of $\sum_i Y_i$ should be close (in TV distance) to the same Poisson distribution, and hence to the distribution of $\sum_i X_i$.
- **The trouble**: approximation by Poisson distribution works well only when the p_i's are relatively small.

▷ The approach:

- Use translated Poisson distributions for those p_i's of intermediate values.
- Use Poisson distributions for those p_i's close to 0 or 1.

The translated Poisson distributions

The translated Poisson distributions (TP) [Röllin 2006]

We say that an integer random variable Y has a *translated Poisson distribution* $\mathcal{L}(Y) = TP(\mu, \sigma^2)$ with parameters μ and σ^2 if

$$\mathcal{L}(Y) = \text{Poisson}(\sigma^2 + \{\mu - \sigma^2\}),$$

where $\{\mu - \sigma^2\}$ represents the fractional part of $\mu - \sigma^2$.

Categories of the p_i's

- First, we define the following subintervals of [0,1] (for some $\alpha \in (0,1)$):
  - $\mathcal{L}(z) := [0, \frac{\lfloor z^\alpha \rfloor}{z})$.
  - $\mathcal{M}_1(z) := [\frac{\lfloor z^\alpha \rfloor}{z}, \frac{z/2}{z}]$.
  - $\mathcal{M}_2(z) := [\frac{z/2}{z}, 1 - \frac{\lfloor z^\alpha \rfloor}{z}]$.
  - $\mathcal{H}(z) := [1 - \frac{\lfloor z^\alpha \rfloor}{z}, 1]$.
- Denote by $\mathcal{L}^*(z) := \{i \mid \mathbf{E}[X_i] \in \mathcal{L}(z)\}$
  - Similarly for $\mathcal{M}_1^*(z)$, $\mathcal{M}_2^*(z)$, and $\mathcal{H}^*(z)$).

Some building blocks

Lemma 1

$$\left\| \sum_{i \in \mathcal{L}^*(z)} X_i - \sum_{i \in \mathcal{L}^*(z)} Y_i \right\| \leq \frac{3}{z^{1-\alpha}}.$$

Lemma 2

$$\left\| \sum_{i \in \mathcal{M}_1^*(z)} X_i - \sum_{i \in \mathcal{M}_1^*(z)} Y_i \right\| \leq O(z^{-(\alpha+\beta-1)/2}) + O(z^{-\alpha}) + O(z^{1/2}) + O(z^{-(1-\beta)}),$$

for some $\beta \in (0,1)$ such that $\alpha + \beta > 1$.

- Symmetric arguments for $\mathcal{M}_2^*(z)$ and $\mathcal{H}^*(z)$.

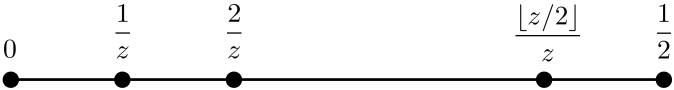
Putting everything together...

Suppose that the random variables $\{Y_i\}_i$ are mutually independent.

$$\left\| \sum_i X_i - \sum_i Y_i \right\| = O(z^{-(1-\alpha)}) + O(z^{-\frac{\alpha+\beta-1}{2}}) + O(z^{-\alpha}) + O(z^{-1/2}) + O(z^{-(1-\beta)}). \qquad (*)$$

Setting $\alpha = \beta = \frac{3}{4}$, we get $(*) = O(z^{-1/4})$. More delicate arguments establish an $O(z^{-1/2})$ bound.

Categorize the expectations $\{\mathbf{E}[X_i]\}_i$



$I_0 = \left[0, \frac{1}{z}\right)$ $I_1 = \left[\frac{1}{z}, \frac{2}{z}\right)$ $I_{\lfloor z/2 \rfloor} = \left[\frac{\lfloor z/2 \rfloor}{z}, \frac{1}{2}\right]$

$I_j^* = \{i \mid \mathbf{E}[X_i] \in I_j\}$, for $j = 0, \ldots, \lfloor z/2 \rfloor$.

$\mathcal{L}^*(z) \cup \mathcal{M}_1^*(z)$ is partitioned into $I_0^*, I^*1, \ldots, I_{\lfloor z/2 \rfloor}^*$

For $j \in \{0, \ldots, \lfloor z/2 \rfloor\}$ with $I_j^* \neq \emptyset$, let $I_j^* = \{j_1, j_2, \ldots, j_{n_j}\}$.

For all $i \in \{1, \ldots, n_j\}$, let $p_i^j := \mathbf{E}[X_{j_i}]$ and $\delta_i^j := p_i^j - \frac{j}{z}$.

Consider the interval $\mathcal{L}(z) := [0, \frac{\lfloor z^\alpha \rfloor}{z})$

Define Y_i, $i \in \mathcal{L}^*(z)$ via the following iterative procedure.

1. $\epsilon_0 := 0$;
2. for $j \leftarrow 0$ to $\lfloor z^\alpha \rfloor - 1$:
   (a) $S_j := \epsilon_j + \sum_{i=1}^{n_j} \delta_i^j$;
   (b) $m_j := \left\lfloor \frac{S_j}{1/z} \right\rfloor$; $\epsilon_{j+1} := S_j - m_j \cdot \frac{1}{z}$;
   (c) set $q_i^j := \frac{j+1}{z}$ for $i = 1, \ldots, m_j$, and $q_i^j := \frac{j}{z}$ for $i = m_j + 1, \ldots, n_j$;
   (d) for all $i \in \{1, \ldots, n_j\}$, let Y_{j_i} be a $\{0, 1\}$-randmo variable with expectation q_i^j;
3. Suppose that $\{Y_i\}_{i \in \mathcal{L}^*(z)}$ are mutually independent.

⋆ It's easy to see that $\epsilon_j < \frac{1}{z}$ $\forall j$, and $m_j \le n_j$.

Consider the interval $\mathcal{L}(z) := [0, \frac{\lfloor z^\alpha \rfloor}{z})$ (contd.)

- For all j,

$$\sum_{i=1}^{n_j} q_i^j = m_j \frac{j+1}{z} + (n_j - m_j)\frac{j}{z} = n_j \frac{j}{z} + m_j \frac{1}{z} = n_j \frac{j}{z} + S_j - \epsilon_{j+1}$$

$$= n_j \frac{j}{z} + \sum_{i=1}^{n_j} \delta_i^j + \epsilon_j - \epsilon_{j+1}$$

$$= \sum_{i=1}^{n_j} p_i^j + \epsilon_j - \epsilon_{j+1}.$$

Thus,

$$\sum_{j=0}^{\lfloor z^\alpha \rfloor - 1} \sum_{i=1}^{n_j} q_i^j = \sum_{j=0}^{\lfloor z^\alpha \rfloor - 1} \sum_{i=1}^{n_j} p_i^j + \epsilon_0 - \epsilon_{\lfloor z^\alpha \rfloor}.$$

Lemma 1.1

$|\sum_{i \in \mathcal{L}^*(z)} \mathbf{E}[Y_i] - \sum_{i \in \mathcal{L}^*(z)} \mathbf{E}[X_i]| = |\sum_{i \in \mathcal{L}^*(z)} q_i - \sum_{i \in \mathcal{L}^*(z)} p_i| \le \frac{1}{z}$.

Poisson approximations

Lemma 1.2 [Barbour, Holst, Janson 1992]

Let $J_1, \ldots, J_n$ be a sequence of indenpendent random indicators with $\mathbf{E}[J_i] = p_i$. Then

$$\left\| \sum_{i=1}^{n} J_i - \text{Poisson}\left(\sum_{i=1}^{n} p_i \right) \right\| \leq \frac{\sum_{i=1}^{n} p_i^2}{\sum_{i=1}^{n} p_i}.$$

Lemma 1.3

Let $\lambda_1, \lambda_2 \in \mathbb{R}_+ \setminus \{0\}$. Then,

$$\|\text{Poisson}(\lambda_1) - \text{Poisson}(\lambda_2)\| \leq e^{|\lambda_1 - \lambda_2|} - e^{-|\lambda_1 - \lambda_2|}.$$

Proof of Lemma 1

Lemma 1

$$\left\|\sum_{i\in\mathcal{L}^*(z)} X_i - \sum_{i\in\mathcal{L}^*(z)} Y_i\right\| \leq \frac{3}{z^{1-\alpha}}.$$

- By Lemma 1.2 we have

$$\left\|\sum_{i\in\mathcal{L}^*(z)} X_i - \text{Poisson}\left(\sum_{i\in\mathcal{L}^*(z)} p_i\right)\right\| \leq \frac{\sum_{i\in\mathcal{L}^*(z)} p_i^2}{\sum_{i\in\mathcal{L}^*(z)} p_i^2} \leq \frac{z^\alpha}{z}$$

and

$$\left\|\sum_{i\in\mathcal{L}^*(z)} Y_i - \text{Poisson}\left(\sum_{i\in\mathcal{L}^*(z)} q_i\right)\right\| \leq \frac{\sum_{i\in\mathcal{L}^*(z)} q_i^2}{\sum_{i\in\mathcal{L}^*(z)} q_i^2} \leq \frac{z^\alpha}{z}.$$

So,

$$\left\|\sum_{i\in\mathcal{L}^*(z)} X_i - \sum_{i\in\mathcal{L}^*(z)} Y_i\right\| \leq \frac{2}{z^{1-\alpha}} + (e^{1/z} - e^{-1/z}) \leq \frac{3}{z^{1-\alpha}}.$$

Consider the interval $\mathcal{M}_1(z) := \left[\frac{\lfloor z^\alpha \rfloor}{z}, \frac{z/2}{z}\right)$

Define Y_i, $i \in \mathcal{M}_1^*(z)$ via the following iterative procedure.

1. for $j \leftarrow \lfloor z^\alpha \rfloor$ to $\lfloor \frac{z}{2} \rfloor$:
   (a) $S_j := \sum_{i=1}^{n_j} \delta_i^j$;
   (b) $m_j := \left\lfloor \frac{S_j}{1/z} \right\rfloor$;
   (c) set $q_i^j := \frac{j+1}{z}$ for $i = 1, \ldots, m_j$, and $q_i^j := \frac{j}{z}$ for $i = m_j + 1, \ldots, n_j$;
   (d) for all $i \in \{1, \ldots, n_j\}$, let Y_{j_i} be a $\{0, 1\}$-randmo variable with expectation q_i^j;
2. Suppose that $\{Y_i\}_{i \in \mathcal{M}_1^*(z)}$ are mutually independent.

Quality of the rounding procedure

$$\zeta_j := \sum_{i \in I_j^*} \mathbf{E}[X_i] - \sum_{i \in I_j^*} \mathbf{E}[Y_i].$$

Lemma 2.1

For all $j \in \{\lfloor z^\alpha \rfloor, \ldots, \lfloor \frac{z}{2} \rfloor\}$,

(a) $\zeta_j = \sum_{i=1}^{n_j} \delta_i^j - m_j \cdot \frac{1}{z}$.

(b) $0 \leq \zeta_j \leq \frac{1}{z}$.

(c) $\sum_{i \in I_j^*} \mathbf{Var}[X_i] = n_j \frac{j}{z}(1 - \frac{j}{z}) + (1 - \frac{2j}{z}) \sum_{i=1}^{n_j} \delta_i^j - \sum_{i=1}^{n_j} (\delta_i^j)^2$.

(d) $\sum_{i \in I_j^*} \mathbf{Var}[Y_i] = n_j \frac{j}{z}(1 - \frac{j}{z}) + m_j \frac{1}{z}(1 - \frac{2j+1}{z})$.

(e) $\sum_{i \in I_j^*} \mathbf{Var}[X_i] - \sum_{i \in I_j^*} \mathbf{Var}[Y_i] = (1 - \frac{2j}{z})\zeta_j + (m_j \frac{1}{z^2} - \sum_{i=1}^{n_j} (\delta_i^j)^2)$.

Proof of Lemma 2

Lemma 2

$$\left\| \sum_{i \in \mathcal{M}_1^*(z)} X_i - \sum_{i \in \mathcal{M}_1^*(z)} Y_i \right\| \leq O(z^{-(\alpha+\beta-1)/2}) + O(z^{-\alpha}) + O(z^{1/2}) + O(z^{-(1-\beta)}),$$

for some $\beta \in (0,1)$ such that $\alpha + \beta > 1$.

- Let's distinguish two possibilities for $|\mathcal{M}_1^*(z)|$:
  - $|\mathcal{M}_1^*(z)| \leq z^\beta$;
  - $|\mathcal{M}_1^*(z)| > z^\beta$.
  - ⋆ **Recall**: $\beta \in (0,1)$ such that $\alpha + \beta > 1$.

Case 1 of Lemma 2's proof

Lemma 2.2

If $|\mathcal{M}_1^*(z)| \leq z^\beta$, then $\left\|\sum_{i\in\mathcal{M}_1^*(z)} X_i - \sum_{i\in\mathcal{M}_1^*(z)} Y_i\right\| \leq \frac{z^\beta}{z} = \frac{1}{z^{1-\beta}}$.

- Choose a joint distribution on $\{X_i\}_i \cup \{Y_i\}_i$ such that $\Pr[X_i \neq Y_i] \leq \frac{1}{z}$.

Coupling

X, Y: random variables with distribution $\mathbb{P}$ and $\mathbb{Q}$ on Ω respectively.

$\mathbb{W}$: a distribution on $\Omega \times \Omega$ is a **coupling** of $(\mathbb{P}, \mathbb{Q})$ if

- $\forall x \in \Omega$, $\sum_{y\in\Omega} W(x,y) = \mathbb{P}(x)$.
- $\forall y \in \Omega$, $\sum_{x\in\Omega} W(x,y) = \mathbb{Q}(y)$.

The coupling lemma

$||\mathbb{P} - \mathbb{Q}|| \leq \Pr[X \neq Y]$.

Case 2 of Lemma 2's proof

Lemma 2.3

If $|\mathcal{M}_1^*(z)| > z^\beta$, then

$$\left\| \sum_{i \in \mathcal{M}_1^*(z)} X_i - \sum_{i \in \mathcal{M}_1^*(z)} Y_i \right\| \leq O(z^{-\frac{\alpha+\beta-1}{2}}) + O(k^{-\alpha}) + O(k^{-\frac{1}{2}}).$$

Approximations by TPs

Lemma 2.4 [Röllin 2006]

Let $J_1, \ldots, J_n$ be a sequence of indenpendent random indicators with $\mathbf{E}[J_i] = p_i$. Then

$$\left\| \sum_{i=1}^{n} J_i - TP(\mu, \sigma^2) \right\| \leq \frac{\sqrt{\sum_{i=1}^{n} p_i^3(1-p_i)} + 2}{\sum_{i=1}^{n} p_i(1-p_i)}.$$

Lemma 2.5 [Barbour & Lindvall @J. Theoret. Prob. 2006]

Let $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1^2, \sigma_2^2 \in \mathbb{R}_+ \setminus \{0\}$ be such that $\lfloor \mu_1 - \sigma_1^2 \rfloor \leq \lfloor \mu_2 - \sigma_2^2 \rfloor$. Then,

$$\left\| TP(\mu_1, \sigma_1^2) - TP(\mu_2, \sigma_2^2) \right\| \leq \frac{|\mu_1 - \mu_2|}{\sigma_1} + \frac{|\sigma_1^2 - \sigma_2^2| + 1}{\sigma_1^2}.$$

- Let
  - $p_i := \mathbf{E}[X_i]$, $q_i = \mathbf{E}[Y_i]$, $\forall i$;
  - $\mu_1 = \sum_{i \in \mathcal{M}_1^*} p_i$, $\mu_2 = \sum_{i \in \mathcal{M}_1^*} q_i$;
  - $\sigma_1^2 = \sum_{i \in \mathcal{M}_1^*} p_i(1-p_i)$, $\sigma_2^2 = \sum_{i \in \mathcal{M}_1^*} q_i(1-q_i)$;

Lemma 2.6

For any $u \in (0, \frac{1}{2})$ and any set $\{p_i\}_{i \in \mathcal{I}}$, where $p_i \in [u, \frac{1}{2}]$ for all $i \in \mathcal{I}$, then

$$\frac{\sqrt{\sum_{i \in \mathcal{I}} p_i^3(1-p_i)}}{\sum_{i \in \mathcal{I}} p_i(1-p_i)} \leq \frac{1 + 2u + 4u^2 - 8u^3}{\sqrt{16|\mathcal{I}|(1 - u - 4u^2 + 4u^3)}}.$$

Lemma 2.7

For the parameters specified above,

$$\left\| TP(\mu_1, \sigma_1^2) - TP(\mu_2, \sigma_2^2) \right\| \leq O(k^{-\alpha}) + O(k^{-1/2}).$$

Proofs are omitted.

Thank you.