No-Regret Online Learning Algorithms

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Credits for the resource

The slides are based on the lectures of Prof. Luca Trevisan: https://lucatrevisan.github.io/40391/index.html

the lectures of Prof. Shipra Agrawal: https://ieor8100.github.io/mab/

the monograph by Prof. Francesco Orabona: https://arxiv.org/abs/1912.13213

and also Elad Hazan's textbook: *Introduction to Online Convex Optimization, 2nd Edition.*

Outline

Outline

Online Convex Optimization

Goal: Design an algorithm such that

- At discrete time steps $t = 1, 2, \ldots$, output $\boldsymbol{x}_t \in \mathcal{K}$, for each t.
  - $\mathcal{K}$: a convex set of feasible solutions.
- After $\boldsymbol{x}_t$ is generated, a convex cost function $f_t : \mathcal{K} \mapsto \mathbb{R}$ is revealed.
- Then the algorithm suffers the loss $f_t(\boldsymbol{x}_t)$.

And we want to minimize the cost.

The difficulty

- The cost functions f_t is unknown before t.
- $f_1, f_2, \ldots, f_t, \ldots$ are not necessarily fixed.
  - Can be generated dynamically by an adversary.

What's the regret?

- The offline optimum: After T steps,

$$\min_{\boldsymbol{x}\in\mathcal{K}} \sum_{t=1}^{T} f_t(\boldsymbol{x}).$$

- The regret after T steps:

$$\text{regret}_T = \sum_{t=1}^{T} f_t(\boldsymbol{x}_t) - \min_{\boldsymbol{x}\in\mathcal{K}} \sum_{t=1}^{T} f_t(\boldsymbol{x}).$$

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- The rescue: $\text{regret}_T \le o(T)$. $\Rightarrow$ **No-Regret** in average when $T \to \infty$.
  - For example, $\text{regret}_T / T = \frac{\sqrt{T}}{T} \to 0$ when $T \to \infty$.

Prerequisites (1/5)

Diameter

Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a bounded convex and closed set in Euclidean space. We denote by D an upper bound on the diameter of $\mathcal{K}$:

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}, \|\boldsymbol{x} - \boldsymbol{y}\| \leq D.$$

Convex set

A set $\mathcal{K}$ is convex if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}$, we have

$$\forall \alpha \in [0, 1], \alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{y} \in \mathcal{K}.$$

Prerequisites (2/5)

Convex function

A function $f : \mathcal{K} \mapsto \mathbb{R}$ is convex if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}$,

$$\forall \alpha \in [0, 1], f((1 - \alpha)\boldsymbol{x} + \alpha \boldsymbol{y}) \leq (1 - \alpha) f(\boldsymbol{x}) + \alpha f(\boldsymbol{y}).$$

Equivalently, if f is differentiable (i.e., $\nabla f(\boldsymbol{x})$ exists for all $\boldsymbol{x} \in \mathcal{K}$), then f is convex if and only if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}$,

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top (\boldsymbol{y} - \boldsymbol{x}).$$

Prerequisites (3/5)

Theorem [Rockafellar 1970]

Suppose that $f : \mathcal{K} \mapsto \mathbb{R}$ is a convex function and let $x \in \text{int dom}(f)$. If f is differentiable at $\boldsymbol{x}$, then for all $\boldsymbol{y} \in \mathbb{R}^d$,

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle.$$

Subgradient

For a function $f : \mathbb{R}^d \mapsto \mathbb{R}$, $\boldsymbol{g} \in \mathbb{R}^d$ is a subgradient of f at $x \in \mathbb{R}^d$ if for all $\boldsymbol{y} \in \mathbb{R}^d$,

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \langle \boldsymbol{g}, \boldsymbol{y} - \boldsymbol{x} \rangle.$$

Prerequisites (4/5)

Projection

The closest point of $\boldsymbol{y}$ in a convex set $\mathcal{K}$ in terms of norm $\|\cdot\|$:

$$\Pi_{\mathcal{K}}(\boldsymbol{y}) := \arg\min_{\boldsymbol{x} \in \mathcal{K}} \|\boldsymbol{x} - \boldsymbol{y}\|.$$

Pythagoras Theorem

Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a convex set, $\boldsymbol{y} \in \mathbb{R}^d$ and $\boldsymbol{x} = \Pi_{\mathcal{K}}(\boldsymbol{y})$. Then for any $\boldsymbol{z} \in \mathcal{K}$, we have

$$\|\boldsymbol{y} - \boldsymbol{z}\| \geq \|\boldsymbol{x} - \boldsymbol{z}\|.$$

Prerequisites (5/5)

Minimum vs. zero gradient

$$\nabla f(\boldsymbol{x}) = 0 \text{ iff } \boldsymbol{x} \in \arg\min_{\boldsymbol{x}\in\mathbb{R}^d} \{f(\boldsymbol{x})\}.$$

First-order optimality condition (FOO)

Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a convex set, $\boldsymbol{x}^* \in \arg\min_{\boldsymbol{x}\in\mathcal{K}} f(\boldsymbol{x})$. Then for any $\boldsymbol{y} \in \mathcal{K}$ we have

$$\nabla f(\boldsymbol{x}^*)^\top (\boldsymbol{y} - \boldsymbol{x}^*) \geq 0.$$

Convex losses to linear losses

- We have the convex loss function $f_t(\boldsymbol{x}_t)$ at time t.
- Say we have subgradients $\boldsymbol{g}_t$ for each $\boldsymbol{x}_t$.
- $f(\boldsymbol{x}_t) - f(\boldsymbol{u}) \leq \langle \boldsymbol{g}, \boldsymbol{x_t} - \boldsymbol{u} \rangle$ for each $\boldsymbol{u} \in \mathbb{R}^d$.

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- Hence, if we define $\tilde{f}_t(\boldsymbol{x}) := \langle \boldsymbol{g}_t, \boldsymbol{x} \rangle$, then for any $\boldsymbol{u} \in \mathbb{R}^d$,

$$\sum_{t=1}^{T} f_t(\boldsymbol{x}_t) - f(\boldsymbol{u}) \leq \sum_{t=1}^{T} \langle \boldsymbol{g}, \boldsymbol{x_t} - \boldsymbol{u} \rangle = \sum_{t=1}^{T} \tilde{f}_t(\boldsymbol{x}_t) - \tilde{f}(\boldsymbol{u}).$$

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OCO $\to$ OLO.

Outline

Online Gradient Descent (GD)

1. **Input:** convex set $\mathcal{K}$, T, $\boldsymbol{x}_1 \in \mathcal{K}$, step size $\{\eta_t\}$.
2. **for** $t \leftarrow 1$ to T **do**:
   1. Play $\boldsymbol{x}_t$ and observe cost $f_t(\boldsymbol{x}_t)$.
   2. Update and Project:

$$\begin{aligned} \boldsymbol{y}_{t+1} &= \boldsymbol{x}_t - \eta_t \nabla f_t(\boldsymbol{x}_t) \\ \boldsymbol{x}_{t+1} &= \Pi_{\mathcal{K}}(\boldsymbol{y}_{t+1}) \end{aligned}$$

3. **end for**

GD for online convex optimization is of no-regret

Theorem A

Online gradient descent with step size $\{\eta_t = \frac{D}{G\sqrt{t}}, t \in [T]\}$ guarantees the following for all $T \geq 1$:

$$\mathrm{regret}_T = \sum_{t=1}^{T} f_t(\boldsymbol{x}_t) - \min_{\boldsymbol{x}^* \in \mathcal{K}} \sum_{t=1}^{T} f_t(\boldsymbol{x}^*) \leq \frac{3}{2} GD\sqrt{T}.$$

Proof of Theorem A (1/3)

- Let $\boldsymbol{x}^* \in \arg\min_{\boldsymbol{x} \in \mathcal{K}} \sum_{t=1}^{T} f_t(\boldsymbol{x})$.
- Since f_t is convex, we have

$$f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}^*) \leq (\nabla f_t(\boldsymbol{x}_t))^\top (\boldsymbol{x}_t - \boldsymbol{x}^*).$$

- By the updating rule for $\boldsymbol{x}_{t+1}$ and the Pythagorean theorem, we have

$$\|\boldsymbol{x}_{t+1} - \boldsymbol{x}^*\|^2 = \|\Pi_{\mathcal{K}}(\boldsymbol{x}_t - \eta_t \nabla f_t(\boldsymbol{x}_t)) - \boldsymbol{x}^*\|^2 \leq \|\boldsymbol{x}_t - \eta_t \nabla f_t(\boldsymbol{x}_t) - \boldsymbol{x}^*\|^2.$$

Proof of Theorem A (2/3)

- Hence

$$\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 \leq \|\mathbf{x}_t - \mathbf{x}^*\|^2 + \eta_t^2 \|\nabla f_t(\mathbf{x}_t)\|^2 - 2\eta_t (\nabla f_t(\mathbf{x}_t))^\top (\mathbf{x}_t - \mathbf{x}^*)$$

$$2(\nabla f_t(\mathbf{x}_t))^\top (\mathbf{x}_t - \mathbf{x}^*) \leq \frac{\|\mathbf{x}_t - \mathbf{x}^*\|^2 - \|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2}{\eta_t} + \eta_t G^2.$$

- Summing above inequality from $t = 1$ to T and setting $\eta_t = \frac{D}{G\sqrt{t}}$ and $\frac{1}{\eta_0} := 0$ we have :

Proof of Theorem A (3/3)

$$
\begin{aligned}
2\left(\sum_{t=1}^{T} f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}^*)\right) &\leq 2\sum_{t=1}^{T}(\nabla f_t(\boldsymbol{x}_t))^\top(\boldsymbol{x}_t - \boldsymbol{x}^*) \\
&\leq \sum_{t=1}^{T} \frac{\|\boldsymbol{x}_t - \boldsymbol{x}^*\|^2 - \|\boldsymbol{x}_{t+1} - \boldsymbol{x}^*\|^2}{\eta_t} + G^2 \sum_{t=1}^{T} \eta_t \\
&\leq \sum_{t=1}^{T} \|\boldsymbol{x}_t - \boldsymbol{x}^*\|^2 \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right) + G^2 \sum_{t=1}^{T} \eta_t \\
&\leq D^2 \sum_{t=1}^{T} \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right) + G^2 \sum_{t=1}^{T} \eta_t \\
&\leq D^2 \frac{1}{\eta_T} + G^2 \sum_{t=1}^{T} \eta_t \\
&\leq 3DG\sqrt{T}.
\end{aligned}
$$

The Lower Bound

Theorem B

Let $\mathcal{K} = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_\infty \leq r\}$ be a convex subset of $\mathbb{R}^d$. Let A be any algorithm for Online Convex Optimization on $\mathcal{K}$. Then for any $T \geq 1$, there exists a sequence of vectors $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_T$ with $\|\boldsymbol{g}_t\|_2 \leq L$ and $\boldsymbol{u} \in \mathcal{K}$ such that the regret of A satisfies

$$\text{regret}_T(\boldsymbol{u}) = \sum_{t=1}^{T} \langle \boldsymbol{g}_t, \boldsymbol{x}_t \rangle - \sum_{t=1}^{T} \langle \boldsymbol{g}_t, \boldsymbol{u} \rangle \geq \frac{\sqrt{2} L D \sqrt{T}}{4}.$$

- The diameter D of $\mathcal{K}$ is at most $\sqrt{\sum_{i=1}^{d} (2r)^2} \leq 2r\sqrt{d}$.
- $\|\boldsymbol{x}\|_\infty \leq r \Leftrightarrow |\boldsymbol{x}(i)| \leq r$ for each $i \in [n]$.

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  - $\boldsymbol{x}_t = (\boldsymbol{x}_t(1), \boldsymbol{x}_t(2), \ldots, \boldsymbol{x}_t(n))$, where $\boldsymbol{x}_t(i) \in [0, 1]$ and $\sum_i \boldsymbol{x}_t(i) = 1$.

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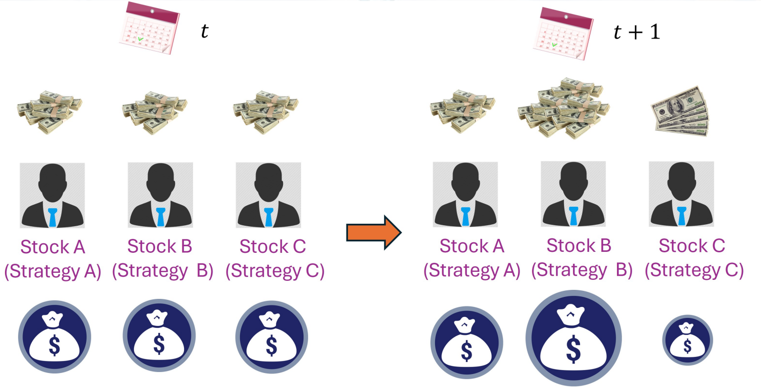
- The loss of following expert i at time t: $\ell_t(i)$.
- The expected loss of the algorithm at time t:

$$\langle \boldsymbol{x}_t, \ell_t \rangle = \sum_{i=1}^{n} \boldsymbol{x}_t(i)\ell_t(i).$$

MWU in Portfolio Rebalancing



The regret of listening to the experts...

$$\mathrm{regret}_T^* = \sum_{t=1}^{T} \langle \boldsymbol{x}_t, \ell_t \rangle - \min_i \sum_{t=1}^{T} \ell_t(i).$$

- The set of feasible solutions $K = \Delta \subseteq \mathbb{R}^n$, probability distributions over $\{1, \ldots, n\}$.
- $f_t(\boldsymbol{x}) = \sum_i \boldsymbol{x}(i)\ell_t(i)$: linear function.

⋆ Assume that $|\ell_t(i)| \leq 1$ for all t and i.

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Multiplicative Weight Update (MWU)

- Maintain a vector of weights $\boldsymbol{w}_t = (\boldsymbol{w}_t(1), \ldots, \boldsymbol{w}_t(n))$ where $\boldsymbol{w}_1 := (1, 1, \ldots, 1)$.
- Update the weights at time t by
  - $\boldsymbol{w}_t(i) := \boldsymbol{w}_{t-1}(i) \cdot e^{-\beta \boldsymbol{\ell}_{t-1}(i)}$.
  - $\boldsymbol{x}_t := \dfrac{\boldsymbol{w}_t(i)}{\sum_{j=1}^{n} \boldsymbol{w}_t(j)}$.

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β: a parameter which will be optimized later.

The weight of expert i at time t: $e^{-\beta \sum_{k=1}^{t-1} \boldsymbol{\ell}_k(i)}$.

MWU is of no-regret

Theorem 1 (MWU is of no-regret)

Assume that $|\ell_t(i)| \leq 1$ for all t and i. For $\beta \in (0, 1/2)$, the regret of MWU after T steps is bounded as

$$\text{regret}_T^* \leq \beta \sum_{t=1}^{T} \sum_{i=1}^{n} \boldsymbol{x}_t(i)\ell_t^2(i) + \frac{\ln n}{\beta} \leq \beta T + \frac{\ln n}{\beta}.$$

In particular, if $T > 4 \ln n$, then

$$\text{regret}_T^* \leq 2\sqrt{T \ln n}$$

by setting $\beta = \sqrt{\frac{\ln n}{T}}$.

Proof of Theorem 1

Let $W_t := \sum_{i=1}^{n} \boldsymbol{w}_t(i)$.

The idea:

- If the algorithm incurs a large loss after T steps, then W_{T+1} is small.
- And, if W_{T+1} is small, then even the best expert performs quite badly.

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Let $L^* := \min_i \sum_{t=1}^{T} \ell_t(i)$.

The proof (contd.)

Lemma 1 (W_{T+1} is SMALL $\Rightarrow L^*$ is LARGE)

$W_{T+1} \geq e^{-\beta L^*}$.

Proof.

Let $j = \arg\min L^* = \arg\min_i \sum_{t=1}^{T} \ell_t(i)$.

$$W_{T+1} = \sum_{i=1}^{n} e^{-\beta \sum_{t=1}^{T} \ell_t(i)} \geq e^{-\beta \sum_{t=1}^{T} \ell_t(j)} = e^{-\beta L^*}.$$

□

The proof (contd.)

Lemma 2 (MWU brings large loss $\Rightarrow W_{T+1}$ is SMALL)

$$W_{T+1} \leq n \prod_{t=1}^{n} (1 - \beta \langle \mathbf{x}_t, \boldsymbol{\ell}_t \rangle + \beta^2 \langle \mathbf{x}_t, \boldsymbol{\ell}_t^2 \rangle),$$

Proof.

Note: $W_1 = n$.

$$\frac{W_{t+1}}{W_t} = \sum_{i=1}^{n} \frac{\mathbf{w}_{t+1}(i)}{W_t} = \sum_{i=1}^{n} \frac{\mathbf{w}_t(i) \cdot e^{-\beta \boldsymbol{\ell}_t(i)}}{W_t}$$

The proof (contd.)

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$$\leq \sum_{i=1}^{n} \boldsymbol{x}_t(i) \cdot (1 - \beta \boldsymbol{\ell}_t(i) + \beta^2 \boldsymbol{\ell}_t^2(i))$$

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The proof (contd.)

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$\square$

The proof (contd.)

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\end{aligned}$$

□

The proof (contd.)

Hence

$$\ln W_{T+1} \leq \ln n - \left(\sum_{i=1}^{T} \beta \langle \boldsymbol{\ell}_t, \boldsymbol{x}_t \rangle \right) + \left(\sum_{i=1}^{T} \beta^2 \langle \boldsymbol{\ell}_t^2, \boldsymbol{x}_t \rangle \right)$$

and $\ln W_{T+1} \geq -\beta L^*$.

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and $\ln W_{T+1} \ge -\beta L^*$.

Thus,

$$\left(\sum_{t=1}^{T} \langle \boldsymbol{\ell}_t, \boldsymbol{x}_t \rangle\right) - L^* \le \frac{\ln n}{\beta} + \beta \sum_{t=1}^{T} \langle \boldsymbol{\ell}_t^2, \boldsymbol{x}_t \rangle.$$

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Take $\beta = \sqrt{\frac{\ln n}{T}}$, we have $\text{regret}_T \leq 2\sqrt{T \ln n}$.

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- First, we assume to make no assumptions on $\mathcal{K}$ and $\{f_t : L \mapsto \mathbb{R}\}$.
- At time t, we are given previous cost functions $f_1, \ldots, f_{t-1}$, and then give the solution

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  That is, the best solution for the previous $t - 1$ steps.

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- First, we assume to make no assumptions on $\mathcal{K}$ and $\{f_t : L \mapsto \mathbb{R}\}$.
- At time t, we are given previous cost functions $f_1, \ldots, f_{t-1}$, and then give the solution

$$\boldsymbol{x}_t := \arg\min_{\boldsymbol{x} \in \mathcal{K}} \sum_{k=1}^{t-1} f_k(\boldsymbol{x}).$$

  That is, the best solution for the previous $t-1$ steps.

- It seems reasonable and makes sense, doesn't it?

FTL leads to "overfitting"

t:	1
$\boldsymbol{x}_t$:	$(0.5, 0.5)$
ℓ_t:	$(0, 0.5)$
$f_t(\boldsymbol{x}_t)$:	0.25
$\arg\min_{\boldsymbol{x}} \sum_{k=1}^{t} f_k(\boldsymbol{x})$:	$(1, 0)$

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t:	1	2
$\boldsymbol{x}_t$:	$(0.5, 0.5)$	$(1, 0)$
ℓ_t:	$(0, 0.5)$	$(1, 0)$
$f_t(\boldsymbol{x}_t)$:	0.25	1
$\arg\min_{\boldsymbol{x}} \sum_{k=1}^{t} f_k(\boldsymbol{x})$:	$(1, 0)$	$(0, 1)$

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t:	1	2	3
$\boldsymbol{x}_t$:	$(0.5, 0.5)$	$(1, 0)$	$(0, 1)$
ℓ_t:	$(0, 0.5)$	$(1, 0)$	$(0, 1)$
$f_t(\boldsymbol{x}_t)$:	0.25	1	1
$\arg\min_{\boldsymbol{x}} \sum_{k=1}^{t} f_k(\boldsymbol{x})$:	$(1, 0)$	$(0, 1)$	$(1, 0)$

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t:	1	2	3	4
$\boldsymbol{x}_t$:	$(0.5, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$
ℓ_t:	$(0, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$
$f_t(\boldsymbol{x}_t)$:	0.25	1	1	1
$\arg\min_{\boldsymbol{x}} \sum_{k=1}^{t} f_k(\boldsymbol{x})$:	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$

FTL leads to "overfitting"

t:	1	2	3	4	5
$\boldsymbol{x}_t$:	$(0.5, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$
ℓ_t:	$(0, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$
$f_t(\boldsymbol{x}_t)$:	0.25	1	1	1	1
$\arg\min_{\boldsymbol{x}} \sum_{k=1}^{t} f_k(\boldsymbol{x})$:	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$	$(1, 0)$

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ℓ_t:	$(0, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$	...
$f_t(\boldsymbol{x}_t)$:	0.25	1	1	1	1	...
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optimum loss: $\approx T/2$.

FTL's loss: $\approx T$.

regret: $\approx T/2$ (linear).

Analysis of FTL

Theorem 2 (Analysis of FTL)

For any sequence of cost functions $f_1, \ldots, f_t$ and any number of time steps T, the FTL algorithm satisfies

$$\text{regret}_T \leq \sum_{t=1}^{T} (f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}_{t+1})).$$

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Implication: If $f_t(\cdot)$ is Lipschitz w.r.t. to some distance function $\|\cdot\|$, then $\boldsymbol{x}_t$ and $\boldsymbol{x}_{t+1}$ are close $\Rightarrow \|f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}_{t+1})\|$ can't be too large.

Modify FTL: $\boldsymbol{x}_t$'s shouldn't change too much from step by step.

Proof of Theorem 2

Recall that

$$\mathrm{regret}_T = \sum_{t=1}^{T} f_t(\boldsymbol{x}_t) - \min_{\boldsymbol{x}\in\mathcal{K}} \sum_{t=1}^{T} f_t(\boldsymbol{x})$$

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Prove by induction. $T = 1$: The definition of $\boldsymbol{x}_2$.

Assume that it holds up to T. Then:

$$\sum_{t=1}^{T+1} f_t(\boldsymbol{x}_{t+1}) = \sum_{t=1}^{T} f_t(\boldsymbol{x}_{t+1}) + f_{T+1}(\boldsymbol{x}_{T+2}) \le \sum_{t=1}^{T+1} f_t(\boldsymbol{x}_{T+2}) = \min_{\boldsymbol{x}\in\mathcal{K}} \sum_{t=1}^{T+1} f_t(\boldsymbol{x}),$$

where

$$\sum_{t=1}^{T} f_t(\boldsymbol{x}_{t+1}) \le \min_{\boldsymbol{x}\in\mathcal{K}} \sum_{t=1}^{T} f_t(\boldsymbol{x}) \le \sum_{t=1}^{T} f_t(\boldsymbol{x}_{T+2}).$$

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Outline

Introducing REGULARIZATION

- You might have already been using regularization for quite a long time.

Introducing REGULARIZATION

```
from keras import regularizers
model.add(Dense(64, input_dim=64,
                kernel_regularizer=regularizers.l2(0.01)
```

Introducing REGULARIZATION

```
# l1 data (only 5 informative features)
X_1, y_1 = datasets.make_classification(n_samples=n_samples,
                                        n_features=n_features, n_informative=5,
                                        random_state=1)

# l2 data: non sparse, but less features
y_2 = np.sign(.5 - rnd.rand(n_samples))
X_2 = rnd.randn(n_samples, n_features // 5) + y_2[:, np.newaxis]
X_2 += 5 * rnd.randn(n_samples, n_features // 5)

clf_sets = [(LinearSVC(penalty='l1', loss='squared_hinge', dual=False,
                       tol=1e-3),
             np.logspace(-2.3, -1.3, 10), X_1, y_1),
            (LinearSVC(penalty='l2', loss='squared_hinge', dual=True),
             np.logspace(-4.5, -2, 10), X_2, y_2)]
```

The regularizer

At each step, we compute the solution

$$\boldsymbol{x}_t := \arg\min_{\boldsymbol{x} \in \mathcal{K}} \left(R(\boldsymbol{x}) + \sum_{k=1}^{t-1} f_k(\boldsymbol{x}) \right).$$

This is called Follow the Regularized Leader (FTRL).

In short,

$$\text{FTRL} = \text{FTL} + \text{Regularizer}.$$

Analysis of FTRL

Theorem 3 (Analysis of FTRL)

For

- every sequence of cost function $\{f_t(\cdot)\}_{t\geq 1}$ and
- every regularizer function $R(\cdot)$,

for every $\boldsymbol{x}$, the regret with respect to $\boldsymbol{x}$ after T steps of the FTRL algorithm is bounded as

$$\text{regret}_T(\boldsymbol{x}) \leq \left(\sum_{t=1}^{T} f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}_{t+1})\right) + R(\boldsymbol{x}) - R(\boldsymbol{x}_1),$$

where $\text{regret}_T(\boldsymbol{x}) := \sum_{t=1}^{T}(f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}))$.

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  - We run the FTL algorithm for $T + 1$ steps.
  - The sequence of cost functions: R, f_1, f_2, . . . , f_T.
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  - The solutions: $\boldsymbol{x}_1$, $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, . . . , $\boldsymbol{x}_T$.

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- The regret:

$$R(\boldsymbol{x}_1) - R(\boldsymbol{x}) + \sum_{t=1}^{T} (f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}))$$

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minimizer of $R(\cdot)$

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output of FTL at $t+1$

Outline

Using negative-entropy regularization

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$$R(\boldsymbol{x}) := c \cdot \sum_{i=1}^{n} \boldsymbol{x}(i) \ln \boldsymbol{x}(i).$$

- So our FTRL gives

$$\boldsymbol{x}_t = \arg\min_{\boldsymbol{x} \in \Delta} \left(\sum_{k=1}^{t-1} \langle \ell_k, \boldsymbol{x} \rangle + c \cdot \sum_{i=1}^{n} \boldsymbol{x}(i) \ln \boldsymbol{x}(i) \right).$$

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- So we use Lagrange multiplier to solve

$$\mathcal{L} = \left(\sum_{k=1}^{t-1} \langle \boldsymbol{\ell}_k, \boldsymbol{x} \rangle \right) + c \cdot \left(\sum_{i=1}^{n} \boldsymbol{x}(i) \ln \boldsymbol{x}(i) \right) + \lambda \cdot (\langle \boldsymbol{x}, \mathbf{1} \rangle - 1).$$

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- The partial derivative $\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}(i)}$:

$$\left(\sum_{k=1}^{t-1} \boldsymbol{\ell}_k(i) \right) + c \cdot (1 + \ln \boldsymbol{x}_i) + \lambda$$

Rediscover MWU?

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}(i)} = 0 \quad \Rightarrow \quad \boldsymbol{x}(i) = \exp\left(-1 - \frac{\lambda}{c} - \frac{1}{c}\sum_{k=1}^{t-1} \ell_k(i)\right)$$

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Exactly the solution of MWU if we take $c = 1/\beta$!

Outline

L2 Regularization

- Let's try to apply the FTRL to the case that the regularizer is of L2 norm!
- Consider also linear cost functions but $\mathcal{K} = \mathbb{R}^n$ first.
- What kind of problem we might encounter?

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- The offline optimum could be $-\infty$.
- FTL will also tend to find a solution of "big" size, too.
- To fight this tendency, it makes sense to use a regularizer which penalizes the size of a solution.

$$R(\boldsymbol{x}) := c\|\boldsymbol{x}\|^2.$$

The regularizer of 2-norm tells us...

- $\boldsymbol{x}_1 = \mathbf{0}$.
- $\boldsymbol{x}_{t+1} = \arg\min_{\boldsymbol{x}\in\mathbb{R}^n} c\|\boldsymbol{x}\|^2 + \sum_{k=1}^{t}\langle \boldsymbol{\ell}_k, \boldsymbol{x}\rangle$.
- Compute the gradient:

$$
\begin{aligned}
& 2c\boldsymbol{x} + \sum_{k=1}^{t} \boldsymbol{\ell}_k = 0 \\
\Rightarrow \quad & \boldsymbol{x} = -\frac{1}{2c}\sum_{k=1}^{t} \boldsymbol{\ell}_k .
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$\rightarrow$ penalize the experts that performed badly in the past!

The regret of FTRL with 2-norm regularization

- First, we have

$$f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}_{t+1}) = \langle \ell_t, \boldsymbol{x}_t - \boldsymbol{x}_{t+1} \rangle = \left\langle \ell_t, \frac{1}{2c}\ell_t \right\rangle = \frac{1}{2c}\|\ell_t\|^2.$$

- So, with respect to a solution $\boldsymbol{x}$,

$$\begin{aligned} \text{regret}_T(\boldsymbol{x}) &\leq R(\boldsymbol{x}) - R(\boldsymbol{x}_1) + \sum_{t=1}^{T} f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}_{t+1}) \\ &= c\|\boldsymbol{x}\|^2 + \frac{1}{2c}\sum_{t=1}^{T}\|\ell_t\|^2. \end{aligned}$$

- Suppose that $\|\ell_t\| \leq L$ for each t and $\|\boldsymbol{x}\| \leq D$. Then by optimizing $c = \sqrt{\frac{T}{2D^2L^2}}$, we have

$$\text{regret}_T(\boldsymbol{x}) \leq DL\sqrt{2T}.$$

Dealing with constraints

- Let's deal with the constraint that $\mathcal{K}$ is an arbitrary convex set instead of $\mathbb{R}^n$.
- Using the same regularizer, we have our FTRL which gives

$$\boldsymbol{x}_1 = \arg\min_{\boldsymbol{x}\in\mathcal{K}} c\|\boldsymbol{x}\|^2,$$

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- **The idea:** First solve the unconstrained optimization and then project the solution on K.

Unconstrained optimization + projection

$$\boldsymbol{y}_{t+1} = \arg\min_{\boldsymbol{y}\in\mathbb{R}^n} c\|\boldsymbol{y}\|^2 + \sum_{k=1}^{t}\langle \boldsymbol{\ell}_t, \boldsymbol{y}\rangle.$$

$$\boldsymbol{x}'_{t+1} = \Pi_{\mathcal{K}}(\boldsymbol{y}_{t+1}) = \arg\min_{\boldsymbol{x}\in\mathcal{K}} \|\boldsymbol{x} - \boldsymbol{y}_{t+1}\|.$$

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- **Claim:** $\mathbf{x}'_{t+1} = \mathbf{x}_{t+1}$.

Proof of the claim: $\boldsymbol{x}'_{t+1} = \boldsymbol{x}_{t+1}$

- First, we already have that $\boldsymbol{y}_{t+1} = -\frac{1}{2c}\sum_{k=1}^{t} \ell_t$.
- Then,

$$\begin{aligned} \boldsymbol{x}'_{t+1} &= \arg\min_{\boldsymbol{x}\in\mathcal{K}} \|\boldsymbol{x} - \boldsymbol{y}_{t+1}\| = \arg\min_{\boldsymbol{x}\in\mathcal{K}} \|\boldsymbol{x} - \boldsymbol{y}_{t+1}\|^2 \\ &= \arg\min_{\boldsymbol{x}\in\mathcal{K}} \|\boldsymbol{x}\|^2 - 2\langle \boldsymbol{x}, \boldsymbol{y}_{t+1}\rangle + \|\boldsymbol{y}_{t+1}\|^2 \end{aligned}$$

Proof of the claim: $\boldsymbol{x}'_{t+1} = \boldsymbol{x}_{t+1}$

- First, we already have that $\boldsymbol{y}_{t+1} = -\frac{1}{2c}\sum_{k=1}^{t} \ell_t$.
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$$
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&= \arg\min_{\boldsymbol{x}\in\mathcal{K}} \|\boldsymbol{x}\|^2 - 2\langle \boldsymbol{x}, \boldsymbol{y}_{t+1}\rangle \\
&= \arg\min_{\boldsymbol{x}\in\mathcal{K}} \|\boldsymbol{x}\|^2 - 2\left\langle \boldsymbol{x}, -\frac{1}{2c}\sum_{k=1}^{t} \ell_t \right\rangle \\
&= \arg\min_{\boldsymbol{x}\in\mathcal{K}} c\|\boldsymbol{x}\|^2 + \left\langle \boldsymbol{x}, \sum_{k=1}^{t} \ell_t \right\rangle \\
&= \boldsymbol{x}_{t+1}.
\end{aligned}
$$

To bound the regret

$$f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}_{t+1}) = \langle \boldsymbol{\ell}_t, \boldsymbol{x}_t - \boldsymbol{x}_{t+1} \rangle \quad \leq \quad \|\boldsymbol{\ell}_t\| \cdot \|\boldsymbol{x}_t - \boldsymbol{x}_{t+1}\|$$

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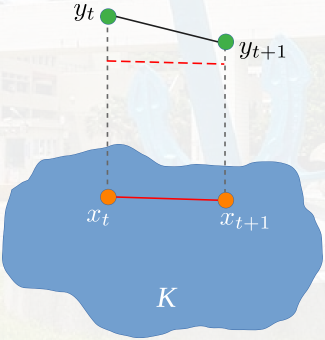

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So, assume $\max_{\boldsymbol{x} \in \mathcal{K}} \|\boldsymbol{x}\| \leq D$ and $\|\boldsymbol{\ell}_t\| \leq L$ for all t, we have

$$\begin{aligned} \text{regret}_T \quad &\leq \quad c\|\boldsymbol{x}^*\|^2 - c\|\boldsymbol{x}_1\|^2 + \frac{1}{2c}\sum_{t=1}^{T} \|\boldsymbol{\ell}_t\|^2 \\ &\leq \quad cD^2 + \frac{1}{2c} TL^2 \end{aligned}$$

To bound the regret

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Outline

Multi-Armed Bandit

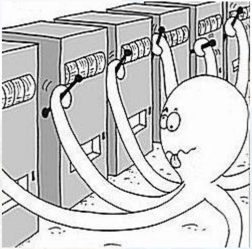

Fig.: Image credit: Microsoft Research

The setting

- We can see N arms as N experts.
- Arms give are independent.
- We can only pull an arm and observe the reward of it.
  - It's NOT possible to observe the reward of pulling the other arms...
- Each arm i has its own reward $r_i \in [0, 1]$.

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- We can see N arms as N experts.
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  - It's NOT possible to observe the reward of pulling the other arms...
- Each arm i has its own reward $r_i \in [0, 1]$.
  - μ_i: the mean of reward of arm i
    - $\hat{\mu}_i$: the empirical mean of reward of arm i
  - μ^*: the mean of reward of the BEST arm.
  - $\Delta_i : \mu^* - \mu_i$.
  - Index of the best arm: $I^* := \arg\max_{i \in \{1,\dots,N\}} \mu_i$.
  - The associated highest expected reward: $\mu^* = \mu_{I^*}$.

The regret formulation for MAB

Let I_t be the arm played by the algorithm at time t.
The regret of the algorithm in T rounds is

$$\mathrm{regret}_T = \sum_{t=1}^{T} (\mu^* - \mu_{I_t})$$

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Outline

A Naïve Greedy Algorithm

Greedy Algorithm

1. For $t \leq cN$, select a random arm with probability $1/N$ and pull it.
2. For $t > cN$, pull the arm $I_t := \arg\max_{i=1,\ldots,N} \hat{\mu}_{i,t}$.

- Here c is a constant.

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  - Arm 2: Fixed reward of $1/4$.
  - After $cN = 2c$ steps, with constant probability, we have $\hat{\mu}_{1,cN} < \hat{\mu}_{2,cN}$.
  - If this is the case, the algorithm will keep pulling arm 2 and will never change!

ϵ-Greedy Algorithm

ϵ-Greedy Algorithm

For all $t = 1, 2, \ldots, N$:

- With probability $1 - \epsilon$, pull arm $I_t := \arg\max_{i=1,\ldots,N} \hat{\mu}_{i,t}$.
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- It looks good.
- Unfortunately, this algorithm still incurs linear regret.
- Indeed,
  - Each arm is pulled in average $\epsilon T/N$ times.
  - Hence the (expected) regret will be at least $\frac{\epsilon T}{N} \sum_{i:\mu_i<\mu^*} \Delta_i$.

Outline

The upper confidence bound algorithm (UCB)

- At each time step (round), we simply pull the arm with the highest "**empirical reward estimate + high-confidence interval size**".
- The empirical reward estimate of arm i at time t:

$$\hat{\mu}_{i,t} = \frac{\sum_{s=1}^{t} I_{s,i} \cdot r_s}{n_{i,t}}.$$

$n_{i,t}$: the number of times arm i is played.
$I_{s,i}$: 1 if the choice of arm is i at time s and 0 otherwise.

- Reward estimate + confidence interval:

$$\mathrm{UCB}_{i,t} := \hat{\mu}_{i,t} + \sqrt{\frac{\ln t}{n_{i,t}}}.$$

Algorithm UCB

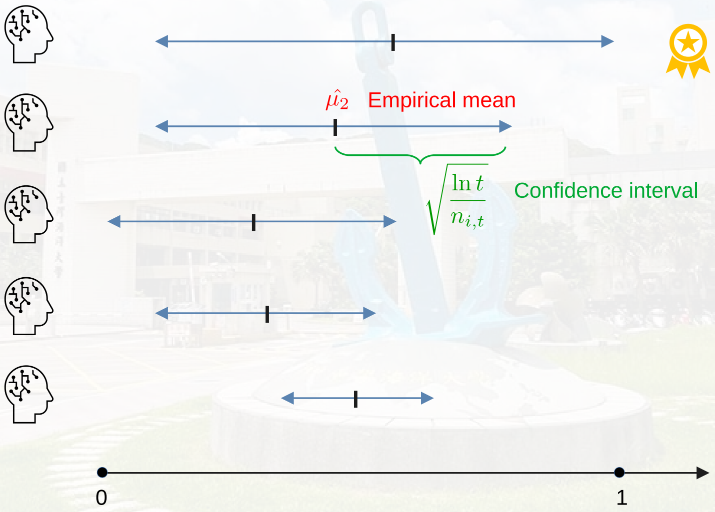
UCB Algorithm

N arms, T rounds such that $T \geq N$.

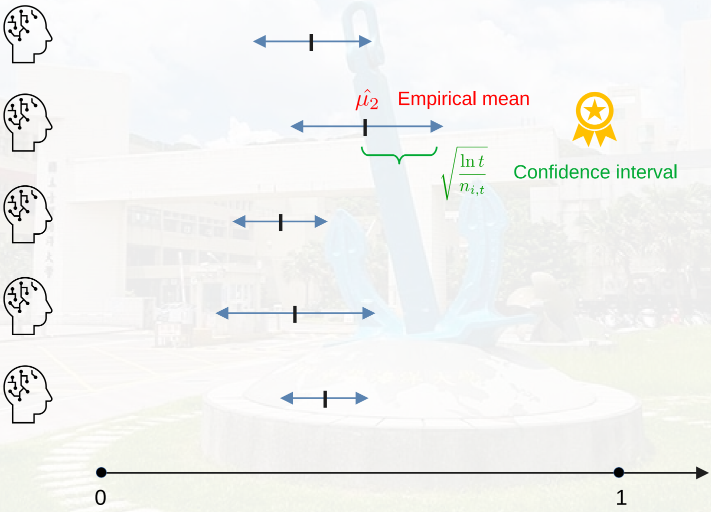
1. For $t = 1, \ldots, N$, play arm t.
2. For $t = N + 1, \ldots, T$, play arm

$$A_t = \arg\max_{i \in \{1,\ldots,N\}} \mathrm{UCB}_{i,t-1}.$$

Algorithm UCB



Algorithm UCB (after more time steps...)



From the Chernoff bound (proof skipped)

For each arm i at time t, we have

$$|\hat{\mu}_{i,t} - \mu_i| < \sqrt{\frac{\ln t}{n_{i,t}}}$$

with probability $\geq 1 - 2/t^2$.

Immediately, we know that

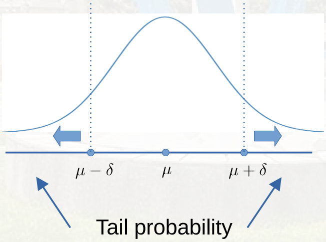
- with prob. $\geq 1 - 2/t^2$, $\mathrm{UCB}_{i,t} := \hat{\mu}_{i,t} + \sqrt{\frac{\ln t}{n_{i,t}}} > \mu_i$.
- with prob. $\geq 1 - 2/t^2$, $\hat{\mu}_{i,t} < \mu_i + \frac{\Delta_i}{2}$ when $n_{i,t} \geq \frac{4 \ln t}{\Delta_i^2}$.

Appendix: Tail probability by the Chernoff/Hoeffding bound

The Chernoff/Hoeffding bound

For independent and identically distributed (i.i.d.) samples $x_1, \ldots, x_n \in [0, 1]$ with $\mathbb{E}[x_i] = \mu$, we have

$$\Pr\left[\left|\frac{\sum_{i=1}^{n} x_i}{n} - \mu\right| \geq \delta\right] \leq 2e^{-2n\delta^2}.$$



Very unlikely to play a suboptimal arm

Lemma 3

At any time step t, if a suboptimal arm i (i.e., $\mu_i < \mu^*$) has been played for $n_{i,t} \geq \frac{4 \ln t}{\Delta_i^2}$ times, then $\text{UCB}_{i,t} < \text{UCB}_{I^*,t}$ with probability $\geq 1 - 4/t^2$. Therefore, for any t,

$$\Pr\left[I_{t+1,i} = 1 \,\middle|\, n_{i,t} \geq \frac{4 \ln t}{\Delta_i^2} \right] \leq \frac{4}{t^2}.$$

Proof of Lemma 3

With probability $< 2/t^2 + 2/t^2$ (union bound) that

$$\begin{aligned} \mathrm{UCB}_{i,t} = \hat{\mu}_{i,t} + \sqrt{\frac{\ln t}{n_{i,t}}} &\leq \hat{\mu}_{i,t} + \frac{\Delta_i}{2} \\ &< \left(\mu_i + \frac{\Delta_i}{2}\right) + \frac{\Delta_i}{2} \\ &= \mu^* < \mathrm{UCB}_{i^*,t} \end{aligned}$$

does NOT hold.

Playing suboptimal arms for very limited number of times

Lemma 4

For any arm i with $\mu_i < \mu^*$,

$$\mathbb{E}[n_{i,T}] \leq \frac{4 \ln T}{\Delta_i^2} + 8.$$

$$\begin{aligned}
\mathbb{E}[n_{i,T}] &= 1 + \mathbb{E}\left[\sum_{t=N}^{T} \mathbb{1}\left\{I_{t+1,i} = 1\right\}\right] \\
&= 1 + \mathbb{E}\left[\sum_{t=N}^{T} \mathbb{1}\left\{I_{t+1,i} = 1, n_{i,t} < \frac{4 \ln t}{\Delta_i^2}\right\}\right] \\
&\quad + \mathbb{E}\left[\sum_{t=N}^{T} \mathbb{1}\left\{I_{t+1,i} = 1, n_{i,t} \geq \frac{4 \ln t}{\Delta_i^2}\right\}\right]
\end{aligned}$$

Proof of Lemma 4 (contd.)

$$
\begin{aligned}
\mathbb{E}[n_{i,T}] &\leq \frac{4\ln T}{\Delta_i^2} + \mathbb{E}\left[\sum_{t=N}^{T} \mathbb{1}\left\{I_{t+1,i} = 1, n_{i,t} \geq \frac{4\ln t}{\Delta_i^2}\right\}\right] \\
&= \frac{4\ln T}{\Delta_i^2} + \sum_{t=N}^{T} \Pr\left[I_{t+1,i} = 1, n_{i,t} \geq \frac{4\ln t}{\Delta_i^2}\right] \\
&= \frac{4\ln T}{\Delta_i^2} + \sum_{t=N}^{T} \Pr\left[I_{t+1,i} = 1 \,\middle|\, n_{i,t} \geq \frac{4\ln t}{\Delta_i^2}\right] \cdot \Pr\left[n_{i,t} \geq \frac{4\ln t}{\Delta_i^2}\right] \\
&\leq \frac{4\ln T}{\Delta_i^2} + \sum_{t=N}^{T} \frac{4}{t^2} \\
&\leq \frac{4\ln T}{\Delta_i^2} + 8
\end{aligned}
$$

Proof of Lemma 4 (contd.)

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\begin{aligned}
\mathbb{E}[n_{i,T}] &\leq \frac{4\ln T}{\Delta_i^2} + \mathbb{E}\left[\sum_{t=N}^{T} \mathbb{1}\left\{I_{t+1,i} = 1, n_{i,t} \geq \frac{4\ln t}{\Delta_i^2}\right\}\right] \\
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&\leq \frac{4\ln T}{\Delta_i^2} + \sum_{t=N}^{T} \frac{4}{t^2} \\
&\leq \frac{4\ln T}{\Delta_i^2} + 8 \ \ (\text{since } \sum_{t=1}^{\infty} 1/t^2 = \pi^2/6).
\end{aligned}
$$

The regret bound for the UCB algorithm

Theorem 4

For all $T \geq N$, the (expected) regret by the UCB algorithm in round T is

$$\mathbb{E}[\text{regret}_T] \leq 5\sqrt{NT \ln T} + 8N.$$

Proof of Theorem 4

- Divide the arms into two groups:
  1. Group ONE (G_1): "almost optimal arms" with $\Delta_i < \sqrt{\frac{N}{T} \ln T}$.
  2. Group TWO (G_2): "bad" arms with $\Delta_i \geq \sqrt{\frac{N}{T} \ln T}$.

$$\sum_{i \in G_1} n_{i,T} \Delta_i \leq \left(\sqrt{\frac{N}{T} \ln T} \right) \sum_{i \in G_1} n_{i,T} \leq T \cdot \sqrt{\frac{N}{T} \ln T} = \sqrt{NT \ln T}.$$

By Lemma 4,

$$\begin{aligned} \sum_{i \in G_2} \mathbb{E}[n_{i,T}] \Delta_i \leq \sum_{i \in G_2} \frac{4 \ln T}{\Delta_i} + 8\Delta_i &\leq \sum_{i \in G_2} 4\sqrt{\frac{T \ln T}{N}} + 8 \\ &\leq 4\sqrt{NT \ln T} + 8N. \end{aligned}$$

Outline

Time Decaying ϵ-Greedy Algorithm

What if the horizon T is known in advance when we run ϵ-Greedy?

Time-Decaying ϵ-Greedy Algorithm

For all $t = 1, 2, \ldots, N$, set $\epsilon := N^{1/3}/T^{1/3}$:

- With probability $1 - \epsilon$, pull arm $I_t := \arg\max_{i=1,\ldots,N} \hat{\mu}_{i,t}$.
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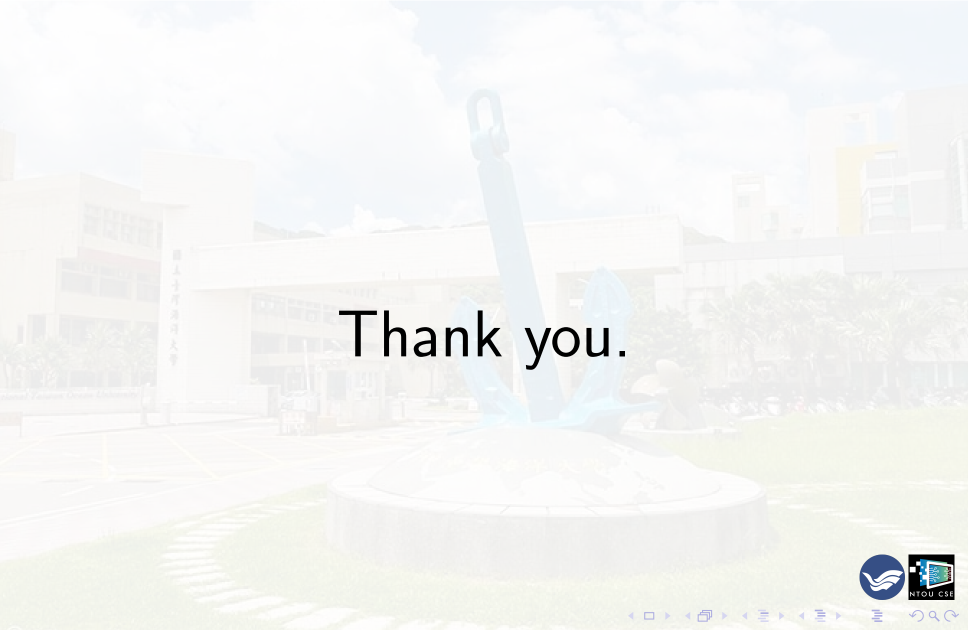
- With probability ϵ, select an arm uniformly at random (i.e., each with probability $1/N$).

Theorem

Time-Decaying ϵ-Greedy Algorithm gets roughly $O(N^{1/3}T^{2/3})$ regret.



Thank you.