Aggregating Partial Rankings with Applications to Peer Grading in Massive Online Open Courses

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Outline

MOOCs

- MOOCs: Massive Online Open Courses
  - e.g., Coursera, EdX.
- Outscourcing the grading task to the students.
- They may have incentives to assign LOW scores to everybody else.
  - ▷ Ask each student to grade a SMALL number of her peers' assignments.
  - Then merge individual rankings into a global one (like social choice).

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Terminologies

- $\mathcal{A}$: universe of n elements (students).
- (n, k)-grading scheme:
  a collection $\mathcal{B}$ of size-k subsets (bundles) of $\mathcal{A}$, such that each element of $\mathcal{A}$ belongs to exactly k subsets of $\mathcal{B}$.
- The bundle graph:
  Represent the (n, k)-grading scheme with a bipartite graph.
- $\prec_b$: a ranking of the element b contains (partial order).

The aggregation rule

An aggregation rule:
profile of partial rankings $\mapsto$ complete ranking of all elements.

- Borda:

SPRING FEAST 2016 BALLOT 春酒地點調查

a	金色三麥	5
b	晶湯匙	3
c	北雲餐廳	1
d	西堤牛排	2
e	卡布里喬莎	4

SPRING FEAST 2016 BALLOT 春酒地點調查

a	金色三麥	5
b	晶湯匙	4
c	北雲餐廳	2
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c	北雲餐廳	1
d	西堤牛排	3
e	卡布里喬莎	2

- a: 14; b: 12; c: 4; d: 6; e: 9.

$$a \prec b \prec e \prec d \prec c.$$

Order-revealing grading scheme

An aggregation rule in peer grading (Borda):

Final Exam Grading

	Curry	2
	Bob	3
	Elvis	1

↓ Alice

Final Exam Grading

	Alice	3
	Curry	2
	David	1

↓ Bob

Final Exam Grading

	Elvis	1
	David	2
	Bob	3

↓ Curry

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	Alice	3
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Final Exam Grading

	Alice	3
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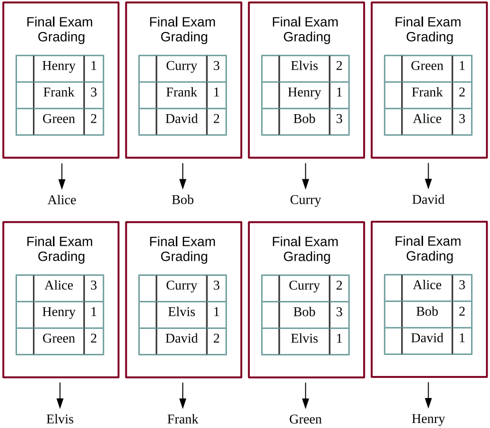
- Alice: 9; Bob: 8; Curry: 5; David: 5; Elvis: 3.

  Alice $\prec$ Bob $\prec$ Curry $\prec$ David $\prec$ Elvis.

Assumption (perfect grading)

Each student grades the assignments in her bundle consistently to the ground truth.

Order-revealing grading scheme (contd.)



- Alice: 9; Bob: 8; Curry: 8; David: 5; Elvis: 4; Frank: 6; Green: 5; Henry: 3.

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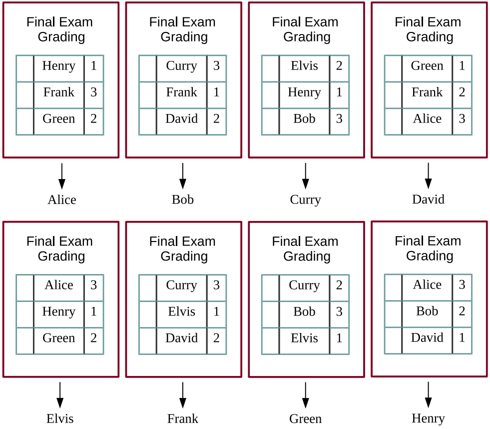

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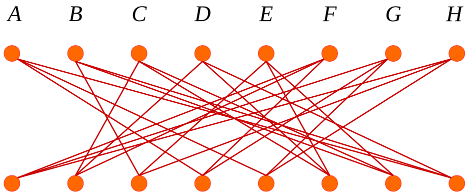
The bundle graph

The bundle graph:



- A random k-regular graph:

  A complete bipartite $K_{n,n} \mapsto$ removing edges $\{v, v\}$, $\forall v \mapsto$ repeat

  > *"draw a perfect matching uniformly at random among all perfect matchings of the remaining graph"*

  for k times.

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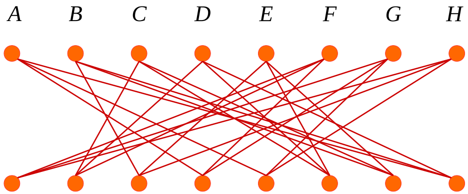

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The limitation on the order revealing scheme

- The property of revealing the ground truth for certain:

$$\forall x, y \in \mathcal{A},\ \exists B \in \mathcal{B} \text{ such that } x, y \in B.$$

  - Suppose NO bundle contains both $x, y \in \mathcal{A}$.
  - Let $\prec, \prec'$ be two complete rankings.
    - x, y are in the first two positions in $\prec, \prec'$;
    - $\prec$ and $\prec'$ differs only in the order of x and y.
  - Clearly, partial rankings within the bundles are identical in both cases.
  - No way to identify whether $\prec$ or $\prec'$ is the ground truth.

- To reveal the ground truth with certainty: $k = \Omega(\sqrt{n})$.
  - $n \cdot \binom{k}{2} \geq \binom{n}{2}$.

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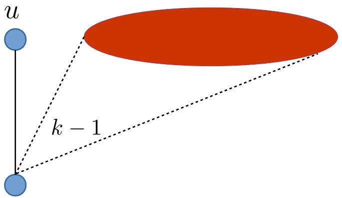
Seeking for approximate order-revealing grading schemes

- Use a bundle graph with a very low degree k (independent of n).
- Randomly permute the elements by $\pi : U \mapsto \mathcal{A}$ before associating them to the nodes of U of the bundle graph.
- Aiming at $\frac{\#\text{correctly recovered pairwise relations}}{\binom{n}{2}}$.

About who grading both u and v

- $\lambda_{u,v} := |N(u) \cap N(v)|$, for $u, v \in U$.
  - $\sum_{v \in U \setminus \{u\}} \lambda_{u,v} = k(k-1)$.



The main result

Theorem 1

When

- Borda is applied as the aggregation rule, and
- all the partial rankings are consistent to the ground truth,

then the expected fraction of correctly recovered pairwise relations is $1 - O(1/\sqrt{k})$.

- $W_{r,q}$: the r.v. denoting $B(a_r) - B(a_q)$ for $r < q$, $a_r, a_q \in \mathcal{A}$.
- $\Gamma^{r,q}_{u,v}$: the event that $\pi(u) = a_r$, $\pi(v) = a_q$.

- $$C := \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \mathbf{E}[\mathbb{1}\{W_{r,q} > 0\}] = \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \Pr[W_{r,q} > 0]$$
$$= \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \left(1 - \sum_{u,v \in U} \Pr[W_{r,q} \leq 0 \mid \Gamma^{r,q}_{u,v}] \cdot \Pr[\Gamma^{r,q}_{u,v}]\right)$$
$$= \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \left(1 - \frac{1}{n(n-1)} \sum_{u,v \in U} \Pr[W_{r,q} \leq 0 \mid \Gamma^{r,q}_{u,v}]\right)$$

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&= \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \left(1 - \sum_{u,v \in U} \Pr[W_{r,q} \le 0 \mid \Gamma_{u,v}^{r,q}] \cdot \Pr[\Gamma_{u,v}^{r,q}]\right) \\
&= \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \left(1 - \frac{1}{n(n-1)} \sum_{u,v \in U} \Pr[W_{r,q} \le 0 \mid \Gamma_{u,v}^{r,q}]\right)
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- Given $\Gamma_{u,v}^{r,q}$,
  - the expected Borda score of a_r is $k + (k(k-1) - \lambda_{u,v}) \cdot \frac{n-r-1}{n-2} + \lambda_{u,v}$.
  - the expected Borda score of a_q is $k + (k(k-1) - \lambda_{u,v}) \cdot \frac{n-q}{n-2}$.
- Thus

$$\mathbf{E}[W_{r.q} \mid \Gamma_{u,v}^{r,q}] = (k(k-1) - \lambda_{u,v})\frac{q-r-1}{n-2} + \lambda_{u,v}.$$

Why?

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Why?

Calculate the Borda score from another point of view

- Element a_r gets one point for each bundle it belongs to;
  - plus one additional point for each appearance of an element with rank higher $> r$ in the bundles a_r belongs to.

- In the bundles of containing a_r:
  - $\lambda_{u,v}$ appearances of a_q in the bundles of a_r.
  - $k(k-1) - \lambda_{u,v}$ appearances of elements different than a_r, a_q.
    - Each of them has prob. $\frac{n-r-1}{n-2}$ to have rank higher than r.
- $\mathbf{E}[B(a_r) \mid \Gamma^{r;q}_{u,v}] = k + (k(k-1) - \lambda_{u,v})\frac{n-r-1}{n-2} + \lambda_{u,v}$.
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Dealing with dependencies

Goal: $\Pr[W_{r,q} \leq 0 \mid \Gamma_{u,v}^{r,q}]$

- Given $\Gamma_{u,v}^{r,q}$, define $S = N(N(u) \cup N(v)) \setminus \{u, v\}$.
- $o : [|S|] \mapsto S$ denotes an arbitrary ordering of nodes of S.
- X_i: the random variable denoting the rank of the element $\pi(o(i))$.
- Define the Doob martingale $Z_0, Z_1, \ldots, Z_{|S|}$ such that
  - $Z_0 = \mathbf{E}[W_{r,q} \mid \Gamma_{u,v}^{r,q}]$;
  - $Z_i = \mathbf{E}[W_{r,q} \mid \Gamma_{u,v}^{r,q}, X_1, \ldots, X_i]$.
- Hence, $W_{r,q} \mid \Gamma_{u,v}^{r,q} = Z_{|S|}$.

Martingale

Martingale

A sequence of random variables $Z_0, Z_1, \ldots, Z_m$ is a martingale w.r.t. a sequence of random variables $X_1, X_2, \ldots, X_m$ if $\forall i = 1, \ldots, m$,

$$\mathbf{E}[Z_i \mid X_1, \ldots, X_{i-1}] = Z_{i-1}.$$

Doob martingale (Joseph L. Doob (1910–2004))

- W: a random variable
- $X_1, \ldots, X_m$: a sequence of m random variables.

The sequence $Z_0, Z_1, \ldots, Z_m$ such that

- $Z_0 = \mathbf{E}[W]$;
- $Z_i = \mathbf{E}[W \mid X_1, \ldots, X_i]$, $\forall i = 1, \ldots, m$

is called a Doob martingale.

Azuma-Hoeffding inequality

Azuma-Hoeffding inequality

Let $Z_0, Z_1, \ldots, Z_m$ be a martingale with $Z_i - Z_{i-1} \leq c_i$ for $i = 1, \ldots, m$. Then, for all $t > 0$,

$$\Pr[Z_m - Z_0 \leq -t] \leq \exp\left(-\frac{t^2}{2\sum_{i=1}^{m} c_i^2}\right).$$

Dealing with dependencies (contd.)

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- Hence, $W_{r,q} \mid \Gamma_{u,v}^{r,q} = Z_{|S|}$.

Lemma 8

$\forall i \in \{1, 2, \ldots, |S|\}$, it holds that $|Z_i - Z_{i-1}| \le 2(\lambda_{u,o(i)} + \lambda_{v,o(i)})$.

Lemma 3

For every k-regular bipartite graph G,

$\theta_{u,v} = 4 \sum_{z \in (N(u) \cup N(v)) \setminus \{u,v\}} (\lambda_{u,z} + \lambda_{v,z})^2 \le 8k(k-1)(4k-3)$.

- Set $t = \mathbf{E}[W_{r,q} \mid \Gamma_{u,v}^{r,q}]$ $(= Z_0)$, by the Azuma-Hoeffding inequality:

$$\begin{aligned}
\Pr[Z_{|S|} - Z_0 \le -t] &= \Pr[W_{r,q} \le 0 \mid \Gamma_{u,v}^{r,q}] \\
&\le \exp\left(-\frac{t^2}{2\sum_{i=1}^{m} c_i^2}\right) \\
&= \exp\left(-\frac{\mathbf{E}[W_{r,q} \mid \Gamma_{u,v}^{r,q}]^2}{2\theta_{u,v}}\right).
\end{aligned}$$

Back to the computation of C

$$
\begin{aligned}
C &= \sum_{r=1}^{n-1}\sum_{q=r+1}^{n}\left(1-\frac{1}{n(n-1)}\sum_{u,v\in U}\Pr[W_{r,q}\le 0 \mid \Gamma_{u,v}^{r,q}]\right)\\
&\ge \sum_{r=1}^{n-1}\sum_{q=r+1}^{n}\left(1-\frac{1}{n(n-1)}\sum_{u,v\in U}\exp\left(-\frac{\mathbf{E}[W_{r,q}\mid \Gamma_{u,v}^{r,q}]^2}{2\theta_{u,v}}\right)\right)\\
&= \sum_{r=1}^{n-1}\sum_{q=r+1}^{n}\left(1-\frac{1}{n(n-1)}\sum_{u,v\in U} e^{-(\beta(u,v)\cdot y(q-r)+\delta(u,v))^2}\right)\\
&= \frac{n(n-1)}{2}-\frac{1}{n(n-1)}\sum_{u,v\in U}\sum_{d=1}^{n-1}(n-d)e^{-(\beta(u,v)\cdot y(d)+\delta(u,v))^2}\\
&\ge \frac{n(n-1)}{2}-\sum_{u,v\in U}\int_0^1(1-y)e^{-(\beta(u,v)\cdot y+\delta(u,v))^2}dy
\end{aligned}
$$

- $\beta(u,v)=\frac{k(k-1)-\lambda_{u,v}}{\sqrt{2\theta_{u,v}}}$;
- $\delta(u,v)=\frac{\lambda_{u,v}}{\sqrt{2\theta_{u,v}}}$;
- $y(t)=\frac{t-1}{n-2}$.

$$
\begin{aligned}
C &\geq \frac{n(n-1)}{2} - \sum_{u,v\in U} \int_0^1 (1-y)e^{-(\beta(u,v)\cdot y+\delta(u,v))^2}dy \\
&\geq \frac{n(n-1)}{2} - \sum_{u,v\in U} \frac{\beta(u,v)+\delta(u,v)}{2\beta(u,v)^2}\sqrt{\pi} \\
&\geq \frac{n(n-1)}{2} - \frac{k-1}{k(k-2)^2}\sqrt{\frac{\pi}{2}} \sum_{u,v\in U} \sqrt{\theta_{u,v}} \\
&\geq \frac{n(n-1)}{2}\left(1 - \frac{48\sqrt{2\pi}}{\sqrt{k}})\right).
\end{aligned}
$$

Claim 9

Let $\beta > 0$, $\delta \geq 0$, then $\int_0^1 (1-y)e^{-(\beta y+\delta)^2}dy \leq \frac{\beta+\delta}{2\beta^2}\sqrt{\pi}$.

- $k \geq 3$ (assumption).

Thank you.

Proof of Lemma 8

Recall:

- Given $\Gamma^{r,q}_{u,v}$, define $S = N(N(u) \cup N(v)) \setminus \{u, v\}$.
- $o : [|S|] \mapsto S$ denotes an arbitrary ordering of nodes of S.
- X_i: the random variable denoting the rank of the element $\pi(o(i))$.
- Define the Doob martingale $Z_0, Z_1, \ldots, Z_{|S|}$ such that
  - $Z_0 = \mathbf{E}[W_{r,q} \mid \Gamma^{r,q}_{u,v}]$;
  - $Z_i = \mathbf{E}[W_{r,q} \mid \Gamma^{r,q}_{u,v}, X_1, \ldots, X_i]$.
- Hence, $W_{r,q} \mid \Gamma^{r,q}_{u,v} = Z_{|S|}$.

Lemma 8

$\forall i \in \{1, 2, \ldots, |S|\}$, it holds that $|Z_i - Z_{i-1}| \leq 2(\lambda_{u,o(i)} + \lambda_{v,o(i)})$.

- $\mu_{u,v,w} = |N(u) \cap N(v) \cap N(w)|$.

Key to the proof of Lemma 8

- The Borda score difference $W_{r,q}$ (conditioned on $\Gamma_{u,v}^{r,q}$):
  - **increases** for each appearance of a_q in the same bundle with a_r;
  - **increases** for each appearance of $\pi(o(j))$ in a bundle containing a_r but NOT a_q provided that $r < \text{rank}(\pi(o(j))) < q$;
  - **increases** for each appearance of $\pi(o(j))$ in a bundle containing BOTH a_r, a_q provided that $r < \text{rank}(\pi(o(j))) < q$;
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$$\begin{aligned} W_{r,q} &= \lambda_{u,v} + \sum_{j=1}^{|S|}(\lambda_{u,o(j)} - \mu_{u,v,o(j)}) \cdot \mathbb{1}\{X_j > r\} + \sum_{j=1}^{|S|} \mu_{u,v,o(j)} \cdot \mathbb{1}\{r < X_j < q\} \\ &- \sum_{j=1}^{|S|}(\lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\}) \\ &= \lambda_{u,v} + \sum_{j=1}^{|S|}(\lambda_{u,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\}). \end{aligned}$$

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\begin{aligned}
W_{r,q} &= \lambda_{u,v} + \sum_{j=1}^{|S|}(\lambda_{u,o(j)} - \mu_{u,v,o(j)}) \cdot \mathbb{1}\{X_j > r\} + \sum_{j=1}^{|S|} \mu_{u,v,o(j)} \cdot \mathbb{1}\{r < X_j < q\} \\
&- \sum_{j=1}^{|S|}(\lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\}) \\
&= \lambda_{u,v} + \sum_{j=1}^{|S|}(\lambda_{u,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\}).
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$$\begin{aligned} W_{r,q} &= \lambda_{u,v} + \sum_{j=1}^{|S|} (\lambda_{u,o(j)} - \mu_{u,v,o(j)}) \cdot \mathbb{1}\{X_j > r\} + \sum_{j=1}^{|S|} \mu_{u,v,o(j)} \cdot \mathbb{1}\{r < X_j < q\} \\ &- \sum_{j=1}^{|S|} (\lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\}) \\ &= \lambda_{u,v} + \sum_{j=1}^{|S|} (\lambda_{u,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\}). \end{aligned}$$

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$$W_{r,q} = \lambda_{u,v} + \sum_{j=1}^{|S|}(\lambda_{u,o(j)} - \mu_{u,v,o(j)}) \cdot \mathbb{1}\{X_j > r\} + \sum_{j=1}^{|S|} \mu_{u,v,o(j)} \cdot \mathbb{1}\{r < X_j < q\}$$

$$- \sum_{j=1}^{|S|}(\lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\})$$

$$= \lambda_{u,v} + \sum_{j=1}^{|S|}(\lambda_{u,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\}).$$

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&- \sum_{j=1}^{|S|}(\lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\}) \\
&= \lambda_{u,v} + \sum_{j=1}^{|S|}(\lambda_{u,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\}).
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&- \sum_{j=1}^{|S|}(\lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\}) \\
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$$
\begin{aligned}
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&- \sum_{j=1}^{|S|} (\lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\}) \\
&= \lambda_{u,v} + \sum_{j=1}^{|S|} (\lambda_{u,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\}).
\end{aligned}
$$

Key to the proof of Lemma 8 (contd.)

$$W_{r,q} = \lambda_{u,v} + \sum_{j=1}^{|S|} (\lambda_{u,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\})$$

$$\begin{aligned} Z_i - Z_{i-1} &= \mathbf{E}[W_{r,q} \mid \Gamma_{u,v}^{r,q}, X_1, \ldots, X_i] - \mathbf{E}[W_{r,q} \mid \Gamma_{u,v}^{r,q}, X_1, \ldots, X_{i-1}] \\ &= \sum_{j=i}^{|S|} \lambda_{u,o(j)} \left(\Pr[X_j > r \mid X_1, \ldots, X_i] - \Pr[X_j > r \mid X_1, \ldots, X_{i-1}] \right) - \\ &\quad \sum_{j=i}^{|S|} \lambda_{v,o(j)} \left(\Pr[X_j > q \mid X_1, \ldots, X_i] - \Pr[X_j > q \mid X_1, \ldots, X_{i-1}] \right). \end{aligned}$$

Key to the proof of Lemma 8 (contd.)

$$
\begin{aligned}
Z_i - Z_{i-1} &= \mathbf{E}[W_{r,q} \mid \Gamma^{r,q}_{u,v}, X_1, \ldots, X_i] - \mathbf{E}[W_{r,q} \mid \Gamma^{r,q}_{u,v}, X_1, \ldots, X_{i-1}] \\
&= \sum_{j=i}^{|S|} \lambda_{u,o(j)} \left(\Pr[X_j > r \mid X_1, \ldots, X_i] - \Pr[X_j > r \mid X_1, \ldots, X_{i-1}] \right) - \\
&\quad \sum_{j=i}^{|S|} \lambda_{v,o(j)} \left(\Pr[X_j > q \mid X_1, \ldots, X_i] - \Pr[X_j > q \mid X_1, \ldots, X_{i-1}] \right).
\end{aligned}
$$

Note that:

$$\Pr[X_j > r \mid X_1, \ldots, X_{i-1}] = \tfrac{x+y}{n-i-1}, \qquad \Pr[X_j > q \mid X_1, \ldots, X_{i-1}] = \tfrac{y}{n-i-1} \quad (i \le j \le |S|)$$

$$\Pr[X_j > r \mid X_1, \ldots, X_i] = \tfrac{x+y-\mathbb{1}\{X_i > r\}}{n-i-2}, \ \Pr[X_j > q \mid X_1, \ldots, X_i] = \tfrac{y-\mathbb{1}\{X_i > q\}}{n-i-2} \quad (i+1 \le j \le |S|)$$

- x: # available ranks from $[n] \setminus \{r, q, X_1, \ldots, X_{i-1}\}$ that are between r and q;
- y: # available ranks from $[n] \setminus \{r, q, X_1, \ldots, X_{i-1}\}$ that are $> q$.

Key to the proof of Lemma 8 (contd.)

$$
\begin{aligned}
Z_i - Z_{i-1} \;=\; & \lambda_{u,o(i)}\left(\mathbb{1}\{X_i > r\} - \frac{x+y}{n-i-1}\right) - \lambda_{v,o(j)}\left(\mathbb{1}\{X_i > q\} - \frac{y}{n-i-1}\right) \\
& + \sum_{j=i+1}^{|S|} \lambda_{u,o(j)}\left(\frac{x+y-\mathbb{1}\{X_i > r\}}{n-i-2} - \frac{x+y}{n-i-1}\right) \\
& + \sum_{j=i+1}^{|S|} \lambda_{v,o(j)}\left(\frac{y-\mathbb{1}\{X_i > q\}}{n-i-2} - \frac{y}{n-i-1}\right)
\end{aligned}
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Note that:

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$$

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- y: # available ranks from $[n] \setminus \{r, q, X_1, \ldots, X_{i-1}\}$ that are $> q$.

Key to the proof of Lemma 8 (contd.)

$$\begin{aligned} Z_i - Z_{i-1} &= \lambda_{u,o(i)} \left(\mathbb{1}\{X_i > r\} - \frac{x+y}{n-i-1} \right) - \lambda_{v,o(j)} \left(\mathbb{1}\{X_i > q\} - \frac{y}{n-i-1} \right) \\ &+ \sum_{j=i+1}^{|S|} \lambda_{u,o(j)} \left(\frac{x+y-\mathbb{1}\{X_i > r\}}{n-i-2} - \frac{x+y}{n-i-1} \right) \\ &+ \sum_{j=i+1}^{|S|} \lambda_{v,o(j)} \left(\frac{y-\mathbb{1}\{X_i > q\}}{n-i-2} - \frac{y}{n-i-1} \right) \end{aligned}$$

Note that:

$$\Pr[X_j > r \mid X_1, \ldots, X_{i-1}] = \tfrac{x+y}{n-i-1}, \qquad \Pr[X_j > q \mid X_1, \ldots, X_{i-1}] = \tfrac{y}{n-i-1} \qquad (i \le j \le |S|)$$

$$\Pr[X_j > r \mid X_1, \ldots, X_i] = \tfrac{x+y-\mathbb{1}\{X_i>r\}}{n-i-2}, \ \Pr[X_j > q \mid X_1, \ldots, X_i] = \tfrac{y-\mathbb{1}\{X_i>q\}}{n-i-2} \quad (i+1 \le j \le |S|)$$

- x: # available ranks from $[n] \setminus \{r, q, X_1, \ldots, X_{i-1}\}$ that are between r and q;
- y: # available ranks from $[n] \setminus \{r, q, X_1, \ldots, X_{i-1}\}$ that are $> q$.

Key to the proof of Lemma 8 (contd.)

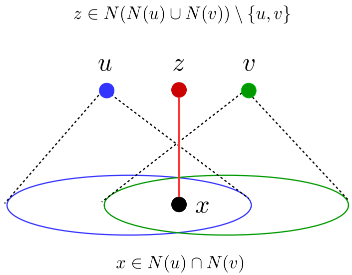
(Assume $n \geq 3k(k-1)+2$; Note: $|S| \leq 2k(k-1)$)

$$\begin{aligned}
Z_i - Z_{i-1} &= \lambda_{u,o(i)}\left(\mathbb{1}\{X_i > r\} - \frac{x+y}{n-i-1}\right) - \lambda_{v,o(j)}\left(\mathbb{1}\{X_i > q\} - \frac{y}{n-i-1}\right) \\
&+ \sum_{j=i+1}^{|S|} \lambda_{u,o(j)}\left(\frac{x+y-\mathbb{1}\{X_i > r\}}{n-i-2} - \frac{x+y}{n-i-1}\right) \\
&+ \sum_{j=i+1}^{|S|} \lambda_{v,o(j)}\left(\frac{y-\mathbb{1}\{X_i > q\}}{n-i-2} - \frac{y}{n-i-1}\right) \\
&= \left(\lambda_{u,o(i)} - \frac{\sum_{j=i+1}^{|S|}\lambda_{u,o(j)}}{n-i-2}\right)\cdot\left(\mathbb{1}\{X_i > r\} - \frac{x+y}{n-i-1}\right) \\
&\quad + \left(\lambda_{v,o(i)} - \frac{\sum_{j=i+1}^{|S|}\lambda_{v,o(j)}}{n-i-2}\right)\cdot\left(\frac{y}{n-i-1} - \mathbb{1}\{X_i > q\}\right).
\end{aligned}$$

Key to the proof of Lemma 3

Lemma 3

For every k-regular bipartite graph G,
$\theta_{u,v} = 4\sum_{z\in(N(u)\cup N(v))\setminus\{u,v\}}(\lambda_{u,z}+\lambda_{v,z})^2 \leq 8k(k-1)(4k-3)$.

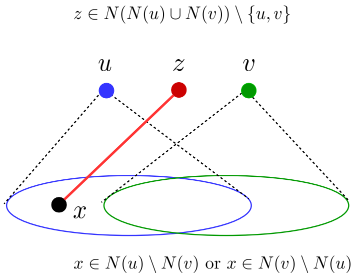


- The edge contributes 2 to the quantity $\lambda_{u,z}+\lambda_{v,z}$.
  - $\lambda_{u,z}+\lambda_{v,z} \leq 2k$.
- The edge contributes $\leq (2k)^2-(2k-2)^2 = 8k-4$ to the quantity $(\lambda_{u,z}+\lambda_{v,z})^2$.
- There are $|N(u)\cap N(v)|(k-2)$ such edges.

Key to the proof of Lemma 3 (contd.)

Lemma 3

For every k-regular bipartite graph G,
$\theta_{u,v} = 4\sum_{z\in(N(u)\cup N(v))\setminus\{u,v\}}(\lambda_{u,z}+\lambda_{v,z})^2 \le 8k(k-1)(4k-3)$.



- The edge contributes 1 to the quantity $\lambda_{u,z} + \lambda_{v,z}$.
  - $\lambda_{u,z} + \lambda_{v,z} \le 2k - 1$.
- The edge contributes $\le (2k-1)^2 - (2k-2)^2 = 4k - 3$ to the quantity $(\lambda_{u,z} + \lambda_{v,z})^2$.
- There are $2(k - |N(u) \cap N(v)|)(k - 1)$ such edges.

Proof of the Gaussian integral (Claim 9)

$$\int_0^1 (1-y)e^{-(\beta y+\delta)^2}dy = \underbrace{\int_0^1 e^{-(\beta y+\delta)^2}dy}_{A} - \underbrace{\int_0^1 ye^{-(\beta y+\delta)^2}dy}_{B}$$

- The error function:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

- The Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}.$$

$$\therefore \quad \int_0^{\infty} e^{-t^2}dt = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad \mathrm{erf}(x) \le \frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} = 1.$$

- $B = \int_0^1 y e^{-(\beta y+\delta)^2} dy$.

  Let $v = \beta y + \delta \quad \therefore \begin{cases} dv = \beta dy \\ y = \frac{v-\delta}{\beta} \end{cases}$.

$$
\begin{aligned}
\therefore B &= \frac{1}{\beta} \int_{\delta}^{\beta+\delta} \frac{v-\delta}{\beta} \cdot e^{-v^2} dv \\
&= \frac{1}{\beta^2} \left(\int_{\delta}^{\beta+\delta} v \cdot e^{-v^2} dv - \delta \cdot \int_{\delta}^{\beta+\delta} e^{-v^2} dv \right) \\
&= \frac{1}{2\beta^2} \int_{\delta}^{\beta+\delta} e^{-v^2} dv^2 - \frac{\delta}{\beta^2} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{2}{\sqrt{\pi}} \int_{\delta}^{\beta+\delta} e^{-v^2} dv \\
&= \frac{1}{2\beta^2} \left(-e^{-(\beta+\delta)^2} + e^{-\delta^2} \right) - \frac{\delta\sqrt{\pi}}{2\beta^2} \left(\mathrm{erf}(\beta+\delta) - \mathrm{erf}(\delta) \right).
\end{aligned}
$$

- $A = \int_0^1 e^{-(\beta y+\delta)^2} dy$.
  Let $u = \beta y + \delta \quad \therefore du = \beta dy$.

$$\begin{aligned} \therefore A &= \frac{1}{\beta} \int_{\delta}^{\beta+\delta} e^{-u^2} du \\ &= \frac{\sqrt{\pi}}{2\beta} \cdot \frac{2}{\sqrt{\pi}} \left(\int_0^{\beta+\delta} e^{-u^2} du - \int_0^{\delta} e^{-u^2} du \right) \\ &= \frac{\sqrt{\pi}}{2\beta} \left(\mathrm{erf}(\beta+\delta) - \mathrm{erf}(\delta) \right). \end{aligned}$$

- Thus

$$\begin{aligned} A - B &= \frac{\beta+\delta}{2\beta^2} \sqrt{\pi} \left(\mathrm{erf}(\beta+\delta) - \mathrm{erf}(\delta) \right) + \frac{1}{2\beta^2} \left(e^{-(\beta+\delta)^2} - e^{-\delta^2} \right) \\ &\leq \frac{\beta+\delta}{2\beta^2} \sqrt{\pi}. \end{aligned}$$