On positive influence dominating sets in social networks

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Outline

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The dominating set problem

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- **Input:** Given a graph $G = (V, E)$ and an integer k.
- **Task:** Find a subset $D \subseteq V$ of size $\leq k$ such that each vertex in $V \setminus D$ is adjacent to (i.e., dominated by) at least one vertex in D.

- A $(1 + O(1))$-approximation algorithm to the dominating set in a *power-law graph* [SODA'2004].
- Another optimization problem:

The Positive Influence Dominating Set problem (PIDS)

- **Input**: Given a graph $G = (V, E)$
- **Task**: Find a subset $D \subseteq V$ such that any $v \in V$ is dominated by at least $\lceil \frac{d(v)}{2} \rceil$ vertices.

Applications of PIDS

- It's helpful for the success of intervention programs for a certain type of social problem (e.g., drinking, smoking, drug related issues...)

⋆ For example, a binge drinker can be converted to an abstainer through intervention programs and have positive impact on his direct friends.

⋆ However, he (she) might turn back into a binge drinker and have negative impact on his (her) direct friends if many of his (her) direct friends are binge drinkers.

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Contribution of this paper

- Explaining the application of PIDS in social networks.
- Showing that PIDS is APX-hard.
  - Though it is still open that whether PIDS is in APX or not.
- Providing a greedy $H(\Delta)$-approximation algorithm for PIDS.
- Conjecture that PIDS is easier in a power-law graph
  - Most social networks follow the power-law.

Outline

Theorem 2.1

PIDS is APX-hard.

Theorem 2.2 (Du & Ko 2000)

The vertex cover problem in cubic graph (VC-cubic) is APX-complete.

We construct an L-reduction from VC-cubic to PIDS.

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APX & L-reduction

APX, APX-hard, & APX-complete

- APX (an abbreviation of "approximable") is the set of **NP** optimization problems that allow constant-factor polynomial-time approximation algorithms.
- A problem $\mathcal{P}$ is APX-complete: $\mathcal{P} \in$ APX & there exists a PTAS-reduction from every problem in APX to $\mathcal{P}$ (APX-hard).

- Note that PTAS = polynomial-time approximation scheme
- Whenever an APX-hard problem is shown to possess a PTAS, every problem in APX has a PTAS.

L-reduction (a kind of PTAS-reduction)

L-reduction

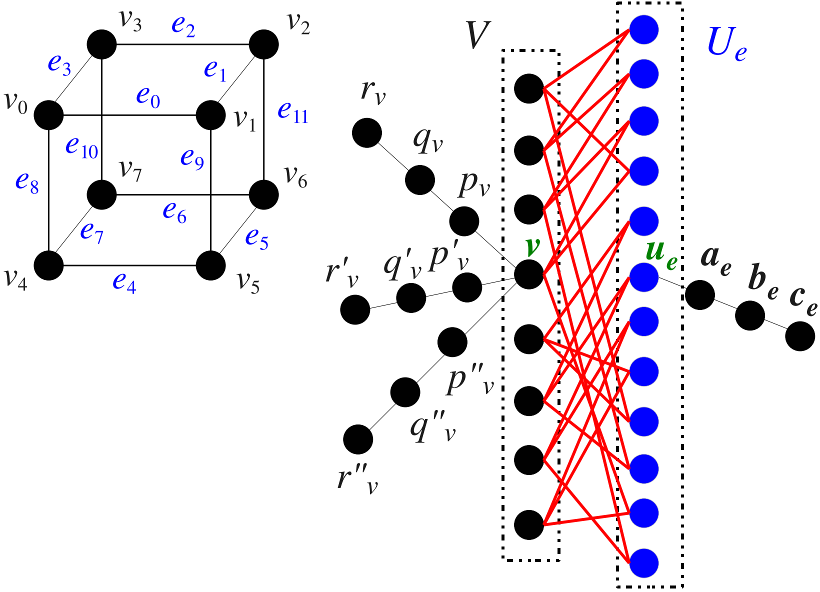
- A and B: two optimization problems;
- c_A and c_B: the cost functions of A and B respectively.
- A pair of functions f and g is an L-reduction if all of the following conditions hold:
  - f and g are computable in polynomial time;
  - if x is an instance of problem A, then $f(x)$ is an instance of problem B;
  - if y is a solution to $f(x)$, then $g(y)$ is a solution to x;
  - there exists $\alpha > 0$ such that

$$|OPT_B(f(x))| \leq \alpha \cdot |OPT_A(x)|;$$

  - there exists $\beta > 0$ such that for every solution y to $f(x)$

$$|OPT_A(x) - c_A(g(y))| \leq \beta \cdot |OPT_B(f(x)) - c_B(y)|.$$

The proof of the APX-hardness of PIDS



The proof of the APX-hardness of PIDS (contd.)

Claim 1

G has a vertex cover of size $\leq k \iff G'$ has a positive influence dominating set of size $\leq k + 9n$ (where $n = |V|$).

- Suppose that G has a vertex cover C of size k.
- Let $D = C \cup \{a_e, b_e \mid e \in E\} \cup \{p_v, q_v, p'_v, q'_v, p''_v, q''_v \mid v \in V\}$.
- Note that $|E| = 3|V|/2$ ($\because$ G is a cubic graph).
- $|D| = |C| + 2 \cdot |E| + 6 \cdot |V| = k + 9n$.
- C is a vertex cover of $G \iff D$ is a PIDS of G'.

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The proof of the APX-hardness of PIDS (contd.)

$opt_{VC-cubic}(G)$: the size of the minimum vertex cover of G;
$opt_{PIDS}(G')$: the size of the minimum PIDS of G'.

- $opt_{VC-cubic}(G) \Longleftrightarrow opt_{PIDS}(G') = opt_{VC-cubic}(G) + 9n$.
- $n/2 = |E|/3 \leq opt_{VC-cubic}(G)$ ($\because \forall v \in V(G)$, $deg(v) = 3$).
- Plugging $n = (opt_{PIDS}(G') - opt_{VC-cubic}(G))/9$ into the inequality, we have

$$opt_{PIDS}(G') \leq 19 \cdot opt_{VC-cubic}(G).$$

- From the proof of Claim 2, we see that if G' has a PIDS, then we can construct in polynomial time, a vertex cover $C = D \cap V$ of G with size $|D| - 9n$.
- Therefore,

$$||C| - opt_{VC-cubic}(G)| = ||D| - opt_{PIDS}(G')|.$$

Outline

- $n_A(v)$: the number of neighbors of v in A for any vertex subset $A \subseteq V$.
- We denote $h(v) = \lceil deg(v)/2 \rceil$.

⋆ $f(A) = \sum_{v \in V} \min(h(v), n_A(v))$.

Important properties of f

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⋆ $f(A) = \sum_{v \in V} \min(h(v), n_A(v))$.

Lemma 3.1

- *$f(\emptyset) = 0$;*
- *$f(A) = \sum_{v \in V} h(v)$ if and only if A is a positive influence dominating set;*
- *If $f(A) < \sum_{v \in V} h(v)$, then there exists a vertex $u \in V \setminus A$ such that $f(A \cup \{u\}) > f(A)$.*

The greedy algorithm

Greedy Algorithm

```
A ← ∅;
while f(A) < ∑_{v∈V} h(v) do
    choose u ∈ V \ A to maximize f(A ∪ {u});
    set A ← A ∪ {u};
end while
output A.
```

A Theorem by Wolsey (*Combinatorica*, 1982)

Theorem 3.2 (Wolsey 1982)

Suppose that f is a monotone increasing, submodular integer function with $f(\emptyset) = 0$. Then the above Greedy Algorithm produces an approximation solution with a factor of $H(\gamma)$ from optimal, where $\gamma = \max_{v \in V} f(\{v\})$ and $H(\gamma) = \sum_{i=1}^{\gamma} \frac{1}{i}$.

Definition 3.3

(1) f is monotone increasing if $A \subset B$ implies $f(A) \leq f(B)$.

(2) f is submodular if for any two subsets A and B,

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B).$$

(1) Suppose $A \subset B$, then $n_A(v) \leq n_B(v)$ for all $v \in V$. Hence

$$\begin{aligned} f(A) &= \sum_{v \in V} \min(h(v), n_A(v)) \\ &\leq \sum_{v \in V} \min(h(v), n_B(v)) \\ &= f(B). \end{aligned}$$

(2)

monotone increasing & submodular (contd.)

(2) f is submodular iff for $u \notin B$, $A \subset B$ implies $\delta_u f(A) \geq \delta_u f(B)$ where $\delta_u f(A) = f(A \cup \{u\}) - f(A)$. [Ding-Zhu Du et al. SODA'2008]

- We have

$$\begin{aligned}\delta_u f(A) &= \sum_{v \in V} [\min(h(v), n_{A \cup \{v\}}(v)) - \min(h(v), n_A(v))],\\ \delta_u f(B) &= \sum_{v \in V} [\min(h(v), n_{B \cup \{v\}}(v)) - \min(h(v), n_B(v))]\end{aligned}$$

- For $u \notin B$ and $A \subset B$, we have

$$n_A(v) \leq n_B(v) \text{ and } n_{A \cup \{u\}}(v) \leq n_{B \cup \{u\}}(v).$$

(2) f is submodular iff for $u \notin B$, $A \subset B$ implies $\delta_u f(A) \geq \delta_u f(B)$ where $\delta_u f(A) = f(A \cup \{u\}) - f(A)$. [Ding-Zhu Du et al. SODA'2008]

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$$
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\delta_u f(B) &= \sum_{v \in V} [\min(h(v), n_{B \cup \{v\}}(v)) - \min(h(v), n_B(v))]
\end{aligned}
$$

- For $u \notin B$ and $A \subset B$, we have

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(2)

Case a. $n_{A\cup\{u\}}(v) \leq h(v)$. In this case,

$$\begin{aligned}
\delta_u f(A) &= \min(h(v), n_{A\cup\{u\}}(v)) - \min(h(v), n_A(v)) \\
&= n_{A\cup\{u\}}(v) - n_A(v) \\
&= n_{B\cup\{u\}}(v) - n_B(v) \\
&\geq \min(h(v), n_{B\cup\{v\}}(v)) - \min(h(v), n_B(v)) \\
&= \delta_u f(B).
\end{aligned}$$

Case b. $n_{A\cup\{u\}}(v) \leq h(v)$. In this case,

$$\begin{aligned}
\delta_u f(A) &= \min(h(v), n_{A\cup\{u\}}(v)) - \min(h(v), n_A(v)) \\
&= 0 \\
&= \min(h(v), n_{B\cup\{u\}}(v)) - \min(h(v), n_B(v)) \\
&= \delta_u f(B).
\end{aligned}$$

Theorem 3.4

The Greedy Algorithm for PIDS produces an approximation solution within a factor of $H(\Delta)$ from optimal where Δ is the maximum vertex degree of the input graph.

Note that $\gamma = \max_{v \in V} f(\{v\}) = \Delta$.

Thank you.