Revenue and Reserve Prices in a Probabilistic Single Item Auction

Noga Alon, Moran Feldman, Moshe Tennenholtz

Algorithmica **Online first** (2015) 1–15.

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19 February 2016

Outline

Motivations

- Real-time bidding in advertising.
  - ad exchanges.
- Publishers (like MSN and Yahoo) attempt to *maximize* the revenue they collect from the advertisers.
  - Doing so by wisely targeting their ads at *right* users.

Probabilistic single-item auction

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- ⋆ P. B. Miltersen & O. Sheffet: **Send mixed signals: earn more, work less.** *EC'12*.

Probabilistic single-item auction (in general):

- Single item, m bidders.
  - The item is chosen randomly from a set of n indivisible goods according to a distribution $p \in \Delta(n)$.
- Second-price auction.
  - Reserve price: a minimum price set by the auctioneer.
  - ⋆ If *no* bid exceeds the reserve price, the item is left *unsold*.
  - ⋆ The player with the highest bid gets the item.
    - The price: *second highest bid* (no less than the reserve price).

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Contribution of this paper

- Extend the previous framework by allowing actions with *reserved prices*.
- Investigate the effect of limiting # different reserve prices on the revenue.
  - Bounding R_ℓ / R_∞.
  - ◇ R_ℓ: the max. possible expected revenue using ℓ different reserve prices.
- Efficient algorithms for computing the optimal set of reserve prices.

The Model

The Model

- n impression types.
- m bidders.
- Each bidder d_j has a value $v(i,j) \geq 0$ for impression type t_i.
- Each impression of type t_i arrives with prob. p_i.
- The auction mechanism can set up to ℓ reserve prices $r_1, r_2, \ldots, r_\ell$.
- Every t_i is assigned $r'_i = \max_{k \in [\ell]}\{r_k \mid r_k \leq v(i,j) \text{ for some } j \in [m]\}$.
  - 0 if there is no such reserve price.
  - ⋆ The auctioneer is familiar with bidders' values.

The Model (contd.)

- Whenever an impression of type t_i arrives, bidders are notified about its exact type and then bidder d_j declares a bid $b(i,j)$.
- d_h, d_s: the bidders with the 1st & the 2nd highest bids, resp.
- The bidder winning the good and the payment are determined by:

◇ if $b(i,h) < r'_i$, no bidder gets the item;

◇ if $b(i,s) < r'_i \leq b(i,h)$, bidder d_h gets the item and pays r'_i;

◇ if $r'_i \leq b(i,s)$, bidder d_h gets the item and pays $b(i,s)$.

◇ Truthful for every given choice of reserve prices.

◇ Declaring $b(i,j) = v(i,j)$ is a weakly dominant strategy for bidder d_j.

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- d_h, d_s: the bidders with the 1st & the 2nd highest bids, resp.
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◇ if $b(i,h) < r_i'$, no bidder gets the item;

◇ if $b(i,s) < r_i' \leq b(i,h)$, bidder d_h gets the item and pays r_i';

◇ if $r_i' \leq b(i,s)$, bidder d_h gets the item and pays $b(i,s)$.

◇ Truthful for every given choice of reserve prices.

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An Example

b: $\begin{pmatrix} 7 & 3 & 1 & 2 \\ 5 & 6 & 5 & 1 \end{pmatrix}$

h: 7 6 5 2

	Value	Reserve Prices
R_1	16/4	$\{5\}$
R_2	18/4	$\{6, 5\}$
R_3	19/4	$\{7, 5, 2\}$
R_∞	20/4	$\{7, 6, 5, 2\}$

- R_ℓ: the expected revenue when the **best** choice of $\leq \ell$ reserve prices are used.
- h_i: the maximal value given by any bidder for impression of type t_i.

Observation 1

$R_\infty = \sum_{i=1}^{n} h_i \cdot p_i$.

An Example

$$\frac{1}{4} \cdot (5+5+5+1) = \frac{16}{4}$$

$$\begin{array}{ll} & \begin{matrix} r'_1 & r'_2 & r'_3 & r'_4 \\ 5 & 5 & 5 & 0 \end{matrix} \\ b: & \begin{pmatrix} 7 & 3 & 1 & 2 \\ 5 & 6 & 5 & 1 \end{pmatrix} \\ h: & \begin{matrix} 7 & 6 & 5 & 2 \end{matrix} \end{array}$$

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Bounds on R_ℓ / R_∞

Table 1 Bounds on R_ℓ / R_∞

	Case	Can be as low as		Is at least	
Uniform probabilities	$\ell \le \ln^{1/2-\varepsilon} n$	ℓ / H_n	(1)	$(1 - o(1)) \cdot \ell / H_n$	(1)
	$\omega(1) \le \ell$	$(1 + o(1)) \cdot c(1 - e^{-1/c})$ where $c = \ell / \ln n$	(2)	$(1 - o(1)) \cdot c(1 - e^{-1/c})$ where $c = \ell / \ln n$	(2)
General probabilities	All	$(1 + o(1)) \cdot \ell / n$	(3)	ℓ / n	(3)

Bounding R_ℓ / R_∞

Case I:

Uniform Probability Distribution over the Impression Types

Lemma 1

Assume uniform probabilities, we have that

$$R_1 \geq R_\infty / H_n,$$

where H_n is the nth harmonic number.

Proof of Lemma 1

- Choose a single reserve price h_i $\Rightarrow$ the auctioneer can get a revenue of $\geq h_i$ from impression types $t_1, t_2, \ldots, t_i$.
  - Total revenue $\geq i \cdot h_i / n$.
- If $i \cdot h_i / n \geq R_\infty / H_n$ for some i, then we are done.
- Assume the contrary then we get:

$$\sum_{i=1}^{n} h_i/n < \sum_{i=1}^{n} R_\infty/(i \cdot H_n) = \frac{R_\infty}{H_n} \cdot \sum_{i=1}^{n} \frac{1}{i} = R_\infty$$

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(contradiction)

- We can change Lemma 1 a little bit...

- Let $\sum_{i=1}^{n'} h_i/n = \hat{R}_{n'}$, where $n' \le n$.
- Assume that $i \cdot h_i/n < \hat{R}_{n'}/H_{n'}$ for all i, then we get:

$$\hat{R}_{n'} = \sum_{i=1}^{n'} h_i/n < \sum_{i=1}^{n'} \hat{R}_{n'}/(i \cdot H_{n'}) = \frac{\hat{R}_{n'}}{H_{n'}} \cdot \sum_{i=1}^{n'} \frac{1}{i} = \hat{R}_{n'}$$

(contradiction)

$\therefore \exists i$ such that

$$\frac{i \cdot h_i}{n} \ge \frac{\sum_{i=1}^{n'} h_i/n}{H_{n'}}.$$

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(contradiction)

$\therefore$ $\exists i$ such that

$$R_1 \geq \frac{i \cdot h_i}{n} \geq \frac{\sum_{i=1}^{n'} h_i/n}{H_{n'}}.$$

- Let $\sum_{i=1}^{n'} h_i/n = \hat{R}_{n'}$, where $n' \leq n$.
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- When ℓ is not too large:

Theorem 1

Assume uniform probabilities. Then, for every $1 \le \ell \le \ln^{1/2-\epsilon} n$ and an arbitrarily small $\epsilon > 0$, it always holds that

$$R_\ell \ge (1 - o(1)) \cdot \ell / H_n \cdot R_\infty.$$

Moreover, there exists an instance with uniform probabilities for which

$$R_\ell \le \ell / H_n \cdot R_\infty.$$

Proof of Theorem 1 (part 1)

- Define $c = \ln^\epsilon n$ (Note: $c^\ell \le n$). $\because \ell \le \ln^{1/2-\epsilon} n$
- If $\sum_{i=1}^{c^\ell} h_i/n \ge R_\infty/c$, then by Lemma 1, R_1 is at least:

$$\begin{aligned} \frac{\sum_{i=1}^{c^\ell} h_i/n}{H_{c^\ell}} &\ge (1-o(1)) \cdot \frac{R_\infty/c}{\ell \ln c} \ge (1-o(1)) \cdot \frac{\ell \cdot R_\infty}{\ln^{1-2\epsilon} n \cdot c \cdot \ln c} \\ &= (1-o(1)) \cdot \frac{\ell \cdot R_\infty}{\ln^{1-\epsilon} n \cdot \ln \ln^\epsilon n} = (1-o(1)) \cdot \frac{\ln^\epsilon n}{\ln \ln^\epsilon n} \cdot \frac{\ell}{\ln n} \cdot R_\infty \\ &\ge (1-o(1)) \cdot \frac{\ell}{H_n} \cdot R_\infty, \end{aligned}$$

- This case is complete because $R_\ell \ge R_1$.
- Consider the case: $\sum_{i=1}^{c^\ell} h_i/n < R_\infty/c$.

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Proof of Theorem 1 (part 1) (contd.)

- Define:

$$r_{k,i} = \begin{cases} h_{\lfloor i \cdot c^{k-1} \rfloor} & \text{if } i \cdot c^{k-1} \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

- The ith set of reserve prices: $\{r_{k,i} \mid 1 \leq k \leq \ell\}$.
- Note that $\sum_{i=1}^{n} r_{1,i} = \sum_{i=1}^{n} h_i = n \cdot R_\infty$.

Proof of Theorem 1 (part 1) (contd.)

$$r_{k,i} = \begin{cases} h_{\lfloor i \cdot c^{k-1} \rfloor} & \text{if } i \cdot c^{k-1} \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

- For every $1 \leq i \leq n$:

$$\frac{\lfloor i \cdot c^0 \rfloor \cdot r_{1,i}}{n} + \sum_{k=2}^{\ell} \frac{[\lfloor i \cdot c^{k-1} \rfloor - \lfloor i \cdot c^{k-2} \rfloor] \cdot r_{k,i}}{n} \leq R_\ell$$

$$\Rightarrow \quad \frac{i \cdot r_{1,i}}{n} + \frac{i \cdot (c-2)}{n} \cdot \sum_{k=2}^{\ell} c^{k-2} \cdot r_{k,i} \leq R_\ell$$

$$\Rightarrow \quad \frac{r_{1,i}}{n} + \frac{c-2}{n} \cdot \sum_{k=2}^{\ell} c^{k-2} \cdot r_{k,i} \leq \frac{R_\ell}{i}$$

$$\therefore \quad R_\infty + \frac{c-2}{n} \cdot \sum_{k=2}^{\ell} c^{k-2} \cdot \sum_{i=1}^{n} r_{k,i} \leq R_\ell \cdot H_n. \qquad (1)$$

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$$\Rightarrow \quad \frac{i \cdot r_{1,i}}{n} + \frac{i \cdot (c-2)}{n} \cdot \sum_{k=2}^{\ell} c^{k-2} \cdot r_{k,i} \leq R_\ell$$

$$\Rightarrow \quad \frac{r_{1,i}}{n} + \frac{c-2}{n} \cdot \sum_{k=2}^{\ell} c^{k-2} \cdot r_{k,i} \leq \frac{R_\ell}{i}$$

$$\therefore \quad R_\infty + \frac{c-2}{n} \cdot \sum_{k=2}^{\ell} c^{k-2} \cdot \sum_{i=1}^{n} r_{k,i} \leq R_\ell \cdot H_n. \qquad (1)$$

Proof of Theorem 1 (part 1) (contd.)

- $\because$ h's are non-increasing, for every $2 \leq k \leq \ell$:

$$\sum_{i=1}^{n} r_{k,i} = \sum_{i=1}^{\lfloor n/c^{k-1} \rfloor} h_{\lfloor i \cdot c^{k-1} \rfloor} \geq \frac{\sum_{i=c^{k-1}}^{n} h_i}{c^{k-1}} = \frac{\sum_{i=1}^{n} h_i - \sum_{i=1}^{c^{k-1}-1} h_i}{c^{k-1}}$$
$$\geq \frac{n \cdot R_\infty - n \cdot R_\infty / c}{c^{k-1}} = n \cdot R_\infty \cdot \frac{1 - 1/c}{c^{k-1}},$$

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Proof of Theorem 1 (part 1)

- Adding Inequality (1) we have

$$R_\infty + (c-2) \cdot \sum_{k=2}^{\ell} c^{k-2} \cdot \left[R_\infty \cdot \frac{1-1/c}{c^{k-1}} \right] \leq R_\ell \cdot H_n$$

$$\begin{aligned}
\Rightarrow \quad & R_\infty + (\ell-1) \cdot (1-2/c)(1-1/c) \cdot R_\infty \leq R_\ell \cdot H_n \\
\Rightarrow \quad & R_\infty + (\ell-1) \cdot (1-2/c)^2 \cdot R_\infty \leq R_\ell \cdot H_n \\
\Rightarrow \quad & R_\infty \cdot \ell \cdot (1-2/c)^2 \leq R_\ell \cdot H_n \\
\Rightarrow \quad & R_\ell \geq (1-2/c)^2 \cdot \frac{\ell}{H_n} \cdot R_\infty = (1-o(1)) \cdot \frac{\ell}{H_n} \cdot R_\infty .
\end{aligned}$$

Proof of Theorem 1 (part 2: the bound is tight)

- Consider an instance:
  - uniform distribution over the impression types;
  - single bidder;
  - value for t_i is $1/i$.
- Clearly,
  - $h_i = 1/i$ for every i.
  - $R_\infty = \sum_{i=1}^{n} (1/i)/n = H_n/n$.
- Let's try to upper bound R_ℓ.

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Proof of Theorem 1 (part 2) (contd.)

- Let $r_1, r_2, \ldots, r_\ell$ be the optimal choice of reserve prices.
  - WLOG, assume that for each i, $r_i = h_j$ for some $1 \leq j \leq n$.
  - Assume that every reserve price is *unique*.
- T_k: a set containing all impression types which yield a revenue of r_k.
  - $R_\ell = (1/n) \cdot \sum_{k=1}^{\ell} |T_k| \cdot r_k$.
- If $r_k = h_i$ for some i, then T_k can contain $\leq i$ elements $1, 1/2, \ldots, 1/i$.
  - $|T_k| \cdot r_k \leq i \cdot (1/i) = 1$.
  - Thus,

$$R_\ell \leq \frac{1}{n} \cdot \sum_{k=1}^{\ell} 1 = \frac{\ell}{n} = \frac{\ell}{H_n} \cdot R_\infty .$$

Proof of Theorem 1 (part 2) (contd.)

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Proof of Theorem 1 (part 2) (contd.)

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  - Thus,

$$R_\ell \leq \frac{1}{n} \cdot \sum_{k=1}^{\ell} 1 = \frac{\ell}{n} = \frac{\ell}{H_n} \cdot R_\infty.$$

- For large values of ℓ:

Theorem 2

Assume uniform probabilities. Then, for every $\omega(1) \leq \ell \leq n$, we have

$$R_\ell \geq (1 - o(1)) \cdot c \left(1 - e^{-1/c}\right) \cdot R_\infty,$$

where $c = \ell / \ln n$.

Moreover, there exists an instance for which

$$R_\ell \leq (1 + o(1)) \cdot \left(1 - e^{-1/c}\right) \cdot R_\infty.$$

Proof of Theorem 2

- Let $b = \left\lceil \ell \left(1 + \frac{\ln \ln n}{\ln n}\right)\right\rceil + 1$.
- Try to bound R_b first.
- Let $B = \{t_i \mid h_i \leq h_1 \cdot e^{(1-b)/c}\}$.
- Total contribution of B to R_∞ is bounded by

$$n \cdot \left(h_1 \cdot e^{(1-b)/c}\right) / n \leq h_1 \cdot e^{-(\ln n + \ln \ln n)} = h_1 \cdot n^{-1} \cdot \ln^{-1} n \leq R_\infty \cdot \ln^{-1} n.$$

- Hence,

$$R_\infty \leq \sum_{i \notin B} h_i / n + R_\infty \cdot \ln^{-1} n$$

$$\Rightarrow \quad R_\infty \leq \frac{\sum_{i \notin B} h_i / n}{1 - \ln^{-1} n}.$$

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Proof of Theorem 2 (contd.)

- x: chosen uniformly at random from $[0, 1]$.
- Define:

$$S_j = \{t_i \notin B \mid h_1 \cdot e^{(2-j-x)/c} \geq h_i > h_1 \cdot e^{(1-j-x)/c}\}.$$

$$r_j := h_1 \cdot e^{(1-j-x)}/c, \text{ for } 1 \leq j \leq b.$$

  - Note: every impression type OUTSIDE B belongs to exactly one S_j.
  - Each $t_i \in S_j$ induces revenue $\geq r_j$.

- Let's define b reserve prices to lower bound R_b.

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$S_j = \{t_i \notin B \mid h_1 \cdot e^{(2-j-x)/c} \geq h_i > h_1 \cdot e^{(1-j-x)/c}\}$.

- Assume that $h_i = h_1 \cdot e^{(1-y_i)/c}$ for some y_i.
  - $t_i \in S_{\lceil y_i \rceil}$ if $x \leq 1 + y_i - \lceil y_i \rceil$, and $t_i \in S_{\lceil y_i \rceil - 1}$ otherwise.
- The expected contribution of t_i to R_b is:

$$\begin{aligned} & \int_0^{1+y_i-\lceil y_i \rceil} h_1 \cdot e^{(1-\lceil y_i \rceil - x)/c} dx + \int_{1+y_i-\lceil y_i \rceil}^{1} h_1 \cdot e^{(2-\lceil y_i \rceil - x)/c} dx \\ = & -h_1 c \cdot e^{(1-\lceil y_i \rceil - x)/c} \Big|_0^{1+y_i-\lceil y_i \rceil} - h_1 c \cdot e^{(2-\lceil y_i \rceil - x)/c} \Big|_{1+y_i-\lceil y_i \rceil}^{1} \\ = & \; h_1 c \cdot e^{-y_i/c} \cdot (e^{1/c} - 1) = h_i \cdot c(1 - e^{-1/c}). \end{aligned}$$

▷ Total expected contribution of $t_i \notin B$ to R_b is $\geq c(1 - e^{-1/c}) \cdot \sum_{i \notin B} h_i / n$.

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Proof of Theorem 2 (contd.)

- Thus, there must exist a set of b reserve prices such that $R_b \geq c(1 - e^{-1/c}) \cdot \sum_{i \notin B} h_i / n$.
- By averaging we get:

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\begin{aligned}
R_\ell &\geq \frac{\ell}{b} \cdot c(1 - e^{-1/c}) \cdot \sum_{i \notin B} h_i / n \\
&\geq \frac{\ell \cdot c(1 - e^{-1/c})}{\lceil \ell(1 + \ln \ln n / \ln n) \rceil + 1} \cdot (1 - \ln^{-1} n) \cdot R_\infty \\
&\geq \frac{\ell \cdot (1 - \ln^{-1} n)}{\ell(1 + \ln \ln n / \ln n) + 2} \cdot c(1 - e^{-1/c} n) \cdot R_\infty \\
&= \frac{1 - o(1)}{1 + o(1) + 2/\ell} \cdot c(1 - e^{-1/c}) \cdot R_\infty .
\end{aligned}
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- We omit the second part of Theorem 2 (similar to that of Theorem 1).

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- We omit the second part of Theorem 2 (similar to that of Theorem 1).

Case II:

General Probability Distributions over the Impression Types

Theorem 3

Assume general probabilities. Then, for every $1 \leq \ell \leq n$, we have

$$R_\ell \geq (\ell/n) \cdot R_\infty.$$

Moreover, there exists an instance for which

$$R_\ell \leq (1 + o(1)) \cdot (\ell/n) \cdot R_\infty.$$

Proof of Theorem 3

- $R_\infty = \sum_{i=1}^{n} p_i \cdot h_i$.
  - ▷ $\exists$ S of size ℓ such that $R_\infty \leq (n/\ell) \cdot \sum_{t_i \in S} p_i \cdot h_i$.
- Choose $\{h_i \mid t_i \in S\}$ as the set of ℓ reserve prices.
  - ▷ $R_\ell \geq \sum_{t_i \in S} p_i \cdot h_i \geq (\ell/n) \cdot R_\infty$.

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- Choose $\{h_i \mid t_i \in S\}$ as the set of ℓ reserve prices.
  - ▷ $R_\ell \ge \sum_{t_i \in S} p_i \cdot h_i \ge (\ell/n) \cdot R_\infty$.

Proof of Theorem 3

- $R_\infty = \sum_{i=1}^{n} p_i \cdot h_i$.
  - ▷ $\exists$ S of size ℓ such that $R_\infty \leq (n/\ell) \cdot \sum_{t_i \in S} p_i \cdot h_i$.
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Proof of Theorem 3 (second part)

- Let $f_0, f_1, f_2, \ldots, f_n$ be a set of values such that:
  - $\forall 1 \le i \le n$, $f(i) = \omega(n) \cdot f(i-1)$.
- Consider the instance with single bidder.
  - The value for impression type t_i is $v(i,1) = 1/f_i$.
  - The prob. of t_i is $p_i := f_i / [\sum_{j=1}^{n} f_j]$.
- $R_\infty = \sum_{i=1}^{n} p_i / f_i = \sum_{i=1}^{n} \frac{f_i}{\sum_{j=1}^{n} f_j} \cdot \frac{1}{f_i} = \frac{n}{\sum_{i=1}^{n} f_i}$.
- Now we consider R_ℓ.

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Proof of Theorem 3 (second part contd.)

- Let $r_1, r_2, \ldots, r_\ell$ be the best set of ℓ (unique) reserve prices.
- WLOG, $r_k = 1/f_i$ for some $i(k)$.
- T_k: a set containing all impression types which contribute r_k to R_ℓ.
- $R_\ell = \sum_{k=1}^{\ell} \left(r_k \cdot \sum_{t_i \in T_k} p_i \right)$.
- Every $t_i \in T_k \setminus \{t_{i(k)}\}$ must have $i < i(k)$.

Proof of Theorem 3 (second part contd.)

- Hence,

$$
\begin{aligned}
r_k \cdot \sum_{t_i \in T_k} p_i &\leq r_k \cdot (p_{i(k)} + n \cdot p_{i(k)-1}) = \frac{p_{i(k)}}{f_{i(k)}} + \frac{n \cdot p_{i(k)-1}}{f_{i(k)}} \\
&= \frac{1}{\sum_{j=1}^{n} f_j} + \frac{n \cdot f_{i(k)-1}/f_{i(k)}}{\sum_{j=1}^{n} f_j} = \frac{1+o(1)}{\sum_{j=1}^{n} f_j}.
\end{aligned}
$$

- Therefore,

$$
R_\ell \leq \sum_{k=1}^{\ell} \left(r_k \cdot \sum_{t_i \in T_k} p_i \right) = \ell \cdot \frac{1+o(1)}{\sum_{j=1}^{n} f_j} = (1+o(1)) \cdot \frac{\ell}{n} \cdot R_\infty .
$$

Computing Optimal Reserve Prices

Theorem 4

The optimal set of reserve prices can be calculated efficiently by dynamic programming of filling up a table of size $n \cdot \ell$.

- $T(n', \ell')$: the optimal set of reserve values where only the types $t_{n-n'+1}, t_{n-n'+2}, \ldots, t_n$ and only ℓ reserve prices are allowed.

⋆ The following discussion focuses on the case of a *single bidder*.

Lemma 3

For every $1 \leq n' \leq n$, $T(n', 1)$ can be efficiently computed.

- Check the values $h_{n-n'+1}, h_{n-n'+2}, \ldots, h_n$.

Lemma 4

For every $1 \leq n' \leq n$ and $1 < \ell' \leq \ell$. Given that $T(n'', \ell - 1)$ is known for every $1 \leq n'' \leq n$, then $T(n', \ell')$ can be efficiently computed.

Illustration of the DP

h_i : 5 3 2 2

$$\begin{pmatrix} 5 & 0 & 1 & 2 \\ 1 & 3 & 2 & 0 \end{pmatrix}$$

ℓ	n' 1	2	3	4
1	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
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- $r_1 \leq r_2 \leq \ldots \leq r_{\ell'}$: the set of optimal reserve prices for the auction represented by $T(n', \ell')$.
- S_k: the set of impression types giving revenue of r_k, $1 \leq k \leq \ell'$.

Illustration of the DP (contd.)

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Consider $T(4,1)$:

- r_1 could only be 5, 3 or 2.
- The corresponding values are 5, 6, and 8.
- So we choose $\{2\}$.

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Consider $T(3,2)$:

- The size of S_2: 1, 2, or 3.
- If $S_2 = \{t_2\}$, then $r_2 = h_2 = 3$.
  - ▷ $\{3\} \cup T(2,1) = \{2,3\}$ (value: 7).
- If $S_2 = \{t_2, t_3\}$, then $r_2 = h_3 = 2$.
  - ▷ $\{2\} \cup T(1,1) = \{2\}$ (value: 6).
- If $S_2 = \{t_2, t_3, t_4\}$, then $r_2 = h_4 = 2$.
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Consider $T(4,2)$:

- The size of S_2: 1, 2, 3, or 4.
- If $S_2 = \{t_1\}$, then $r_2 = h_1 = 5$.
  - ▷ $\{5\} \cup T(3,1) = \{2,5\}$ (value: 11).
- If $S_2 = \{t_1, t_2\}$, then $r_2 = h_2 = 3$.
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