Testing k-colorability

Noga Alon and Michael Krivelevich:
Testing k-colorability. *SIAM J. Discrete Math.* **15** (2002) 211–227.

Speaker: Joseph, Chuang-Chieh Lin

Advisor: Professor Maw-Shang Chang

Computation Theory Laboratory
Dept. Computer Science and Information Engineering
National Chung Cheng University, Taiwan

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Outline

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Introduction (model)

- Graph model: dense graph (adjacency matrix) for $G(V, E)$.
  - undirected, no self-loops, ≤ 1 edge between any $u, v \in V$
  - $|V| = n$ vertices and $|E| = \Omega(n^2)$ edges.
- A graph property:
  - A set of graphs closed under isomorphisms.
- Let $\mathbb{P}$ be a graph property.
  - ϵ-far from satisfying $\mathbb{P}$:
    - $\geq \epsilon n^2$ edges should be deleted or added to let the graph satisfy $\mathbb{P}$

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Introduction (property testing)

- Property testing:
  - it does NOT precisely determine YES or NO for a decision problem;
  - requires sublinear running time
- A property tester for $\mathbb{P}$:
  - A randomized algorithm such that
    - it answers "YES" with probability of $\geq 2/3$ if G satisfies $\mathbb{P}$, and
    - it answers "NO" with probability of $\geq 2/3$ if G is ϵ-far from satisfying $\mathbb{P}$
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- Testing emptiness of a graph
  - Testing H-freeness, where H is an edge.
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  - How can it be done?
- Testing connectivity is trivial (for dense graphs).
  - Why?

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- a (proper) k-coloring: a function $f : V \rightarrow \{1, 2, \ldots, k\}$ such that
  - $f(u) \neq f(v)$ if $(u, v) \in E$.
- Equivalent to a k-partition $(V_1, V_2, \ldots, V_k)$ of V such that for each i, $(u, v) \notin E$ for every $u, v \in V_i$.
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- **NP**-complete for $k \geq 3$
- k-colorability is testable.
  - Hereditary graph property is testable [Alon and Shapira 2008] (by *Szemerédi's regularity Lemma*)
    - Dependency of tower of 2's of height polynomial in $1/\epsilon$.
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- The property tester for k-colorability is very simple.

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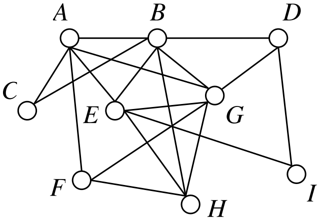
k-coloring-tester (G, s)

Generate a random subset $R \subset V$ of size $s = 36k \ln k/\epsilon^2$
Exhaustively color R by k colors.
Return YES if $G[R]$ is k-colorable, and return NO otherwise.



A	1	2	3	4	5
B	1	2	3	4	5
C	1	2	3	4	5
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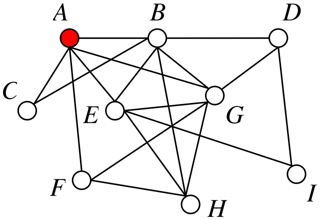

B		2	3	4	5
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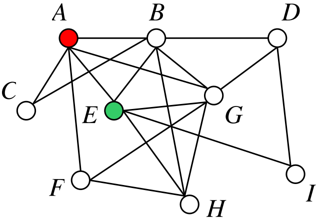

B			3	4	5
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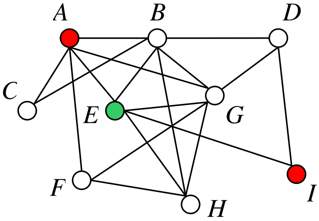

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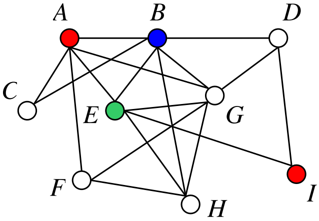

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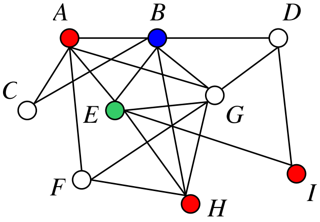
G				4	5
H	1			4	5



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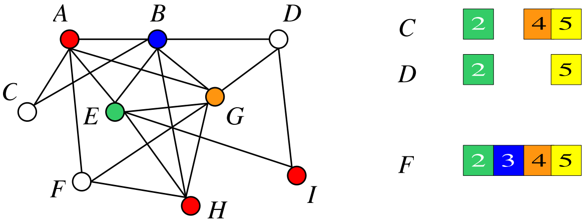
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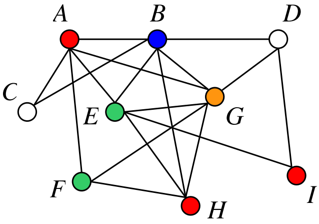
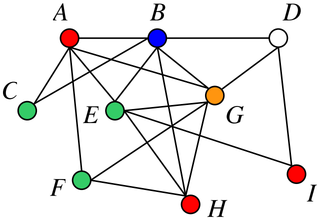
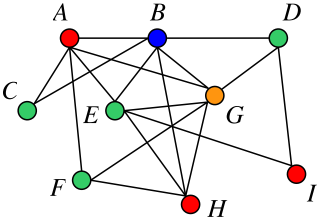


C	2	4	5
D	2		5



D 2 5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Some notations

- Given $S \subseteq V$ and its k-partition $\phi : S \to [k]$.

The list of feasible labels of a vertex $v \in V \setminus S$

$L_\phi(v) = [k] \setminus \{1 \leq i \leq k : \exists u \in S \cap N(v), \phi(u) = i\}$.

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- $S = \{A, B, E, H, I\}$.
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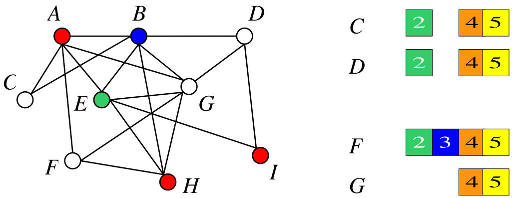

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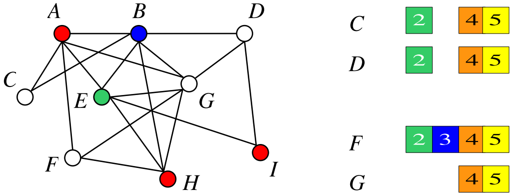

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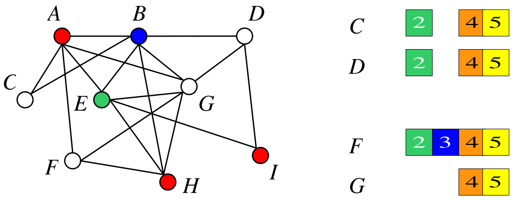

Main idea of the proof

- Assume that G is ϵ-far from being k-colorable.
- Suppose we are given a subset $S \subset R \subset V(G)$ and its k partition $\phi : S \rightarrow [k]$.
- Our aim is to find w.h.p. that:

▷ a succinct (i.e., short & concise) witness in $R \setminus S$ to the fact that ϕ can NOT be extended to a (proper) k-coloring.

  - Witness: a set of vertices which can be used to find out non-k-colorability. (colorless or restricting vertices)
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- If there are a lot of colorless vertices w.r.t. (S, ϕ) ...
  - It is easy to obtain a witness for nonextendability of ϕ.
- What if the number of colorless vertices is small?
  - As G is ϵ-far from being k-colorable, one can show that:
    - ▷ $\exists\ W \subset V$ ($|W|$ is large) s.t. coloring every vertex $v \in W$ by any feasible color w.r.t. ϕ reduces the number of feasible colors of at least $\Omega(\epsilon)n$ neighbors of v.
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- The above process can be represented by an auxiliary tree T.
- Every node of T corresponds to a colorless or a restricting vertex v.
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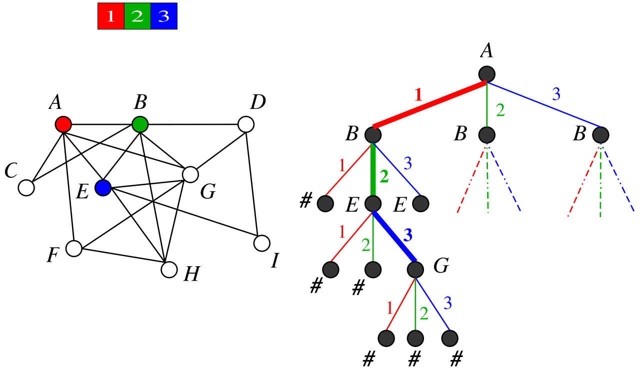

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Outline

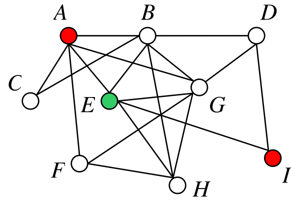
Reducing feasible colors

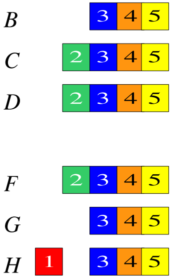
- For every $v \in V \setminus (S \cup U)$:

Estimation of $\#$ excluded feasible colors of $N(v)$ outside $S \cup U$

$$\delta_\phi(v) = \min_{i \in L_\phi(v)} |\{u \in N(v) \setminus (S \cup U) : i \in L_\phi(u)\}|.$$

- U is the set of colorless vertices w.r.t. (S, ϕ).





- $\delta_\phi(B) = \min_{i\in\{3,4,5\}}\{4,4,4\} = 4.$
- $\delta_\phi(C) = \min_{i\in\{2,3,4,5\}}\{0,1,1,1\} = 0.$
- $\delta_\phi(D) = \min_{i\in\{2,3,4,5\}}\{0,2,2,2\} = 0.$
- $\delta_\phi(F) = \min_{i\in\{2,3,4,5\}}\{0,2,2,2\} = 0.$
- $\delta_\phi(G) = \min_{i\in\{3,4,5\}}\{4,4,4\} = 4.$
- $\delta_\phi(H) = \min_{i\in\{1,3,4,5\}}\{0,4,4,4\} = 0.$

Restricting vertices

Restricting vertices

Given a pair (S, ϕ), a vertex is called restricting if $\delta_\phi(v) \geq \epsilon n/2$.

- $W := \{v \in V \setminus (S \cup U) \mid \delta_\phi(v) \geq \epsilon n/2\}$.

An upper bound on the number of monochromatic edges

Claim 1

For every subset $S \subset V$ and every k-partition ϕ of S, to make the graph be k-colorable requires deleting at most $(n-1)(|S|+|U|) + \sum_{v \in V \setminus (S \cup U)} \delta_\phi(v)$ edges.

- "ϵ-far from being k-colorable" makes sense only if $\epsilon n^2 < (n-1)(|S|+|U|) + \sum_{v \in V \setminus (S \cup U)} \delta_\phi(v)$.
- Thus we have the following corollary.

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- Thus we have the following corollary.

Corollary 4.1

If G is ϵ-far from being k-colorable, then for any pair (S, ϕ), where $S \subset V(G)$, $\phi : S \to [k]$, one has

$$\sum_{v \in V \setminus (S \cup U)} \delta_\phi(v) > \epsilon n^2 - n(|S| + |U|),$$

where U is the set of colorless vertices w.r.t. (S, ϕ).

The number of restricting vertices must be large

Claim 2

If G is ϵ-far from being k-colorable, then for any pair (S, ϕ), where $S \subset V(G)$, $\phi : S \to [k]$, one has

$$|U| + |W| > \frac{\epsilon n}{2} - |S|.$$

Proof.

$$\begin{aligned} & \epsilon n^2 - n(|S| + |U|) \\ < & \sum_{v \in V \setminus S \cup U} \delta_\phi(v) \le |W|(n-1) + \sum_{V \setminus (S \cup U \cup W)} \delta_\phi(v) \\ < & |W|n + \frac{\epsilon n^2}{2}. \end{aligned}$$

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Recall the auxiliary tree T for the coloring process

- Consider a leaf t of T.
- $U(t)$: the set of colorless vertices w.r.t. $(S(t), \phi(t))$.
- $W(t)$: the set of restricting vertices w.r.t. $(S(t), \phi(t))$.
- A nonterminal node of T is labeled only when a vertex in $U(t) \cup W(t)$ is chosen.

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An upper bound on the depth of T

Claim 3

The depth of T is bounded by $\frac{2k}{\epsilon}$.

Proof.

- The depth of T is mainly due to the restricting vertices.
- The total length of the lists of feasible colors initially: nk.
- Coloring a vertex $w \in W$: reduces $\geq \epsilon n/2$ colors.
- We cannot make more than $nk/(\epsilon n/2) = 2k/\epsilon$ steps down from the roof of T to a leaf of T.

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$\#$'s and no-proper k-coloring

Claim 4

If a leaf t^* of T is labeled by $\#$, then $\phi(t^*)$ is not a proper k-coloring of $S(t^*)$.

Claim 5

If all leaves t^*'s of T are terminal nodes after j rounds of the algorithm, then the subgraph induced by the labels along the path from the root of T to t^* is not k-colorable.

The leaves of T are all leaves w.h.p. before long

Claim 6

If G is ϵ-far from being k-colorable, then after $36k \ln k/\epsilon^2$ rounds, with probability $\geq 2/3$ all leaves of T are terminal nodes.

Proof.

- T can be embedded into a k-ary tree $T_{k,\frac{2k}{\epsilon}}$ of depth $\frac{2k}{\epsilon}$.
  - $T_{k,\frac{2k}{\epsilon}}$ has at most $1 + k + \ldots + k^{\frac{2k}{\epsilon}} \leq k^{\frac{2k}{\epsilon}+1}$ vertices.
- A round of the algorithm is called successful a colorless vertex or a restricting vertex is picked.

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Proof of Claim 6 (contd.)

Proof.

- Fix some leaf node t of T after $36k \ln k/\epsilon^2$ rounds of the algorithm.
- The total number of successful rounds for the path from the root of T to t is equal to the depth of t.
- Besides, the probability of choosing a colorless or restricting vertex (i.e., $U(t) \cup W(t)$) is at least $\epsilon/2 - S(t)/n = \epsilon/2 - o(1) \geq \epsilon/3$.

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- $\mathbf{Pr}[t \text{ is a nonterminal leaf of } T]$ can be bounded by $\mathbf{Pr}[B(36k \ln k/\epsilon^2, \epsilon/3) < 2k/\epsilon]$.
  - $B(n, p)$ is the Binomial random variable of n Bernoulli trials with probability p of success.
- The Chernoff bound for $B(n, p)$:

$$\mathbf{Pr}[B(m, p) \leq k] \leq \exp\left(-\frac{1}{2p}\frac{(mp-k)^2}{m}\right).$$

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- $\mathbf{Pr}[B(36k \ln k/\epsilon^2, \epsilon/3) < 2k/\epsilon] < k^{-3k/\epsilon}$ by the Chernoff bound.
- Thus by the union bound we conclude that the probability that some node of $T_{k,\frac{2k}{\epsilon}}$ is a nonterminal leaf is $\leq |V(T_{k,\frac{2k}{\epsilon}})| \cdot k^{\frac{-3k}{\epsilon}} < 1/3$.

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- That means, the probability that the algorithm finds a proper k-coloring is less than 1/3.
- Hence we derive the error probability of the algorithm $< 1/3$.

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Thank you!