Exercises of Chapter 1

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Exercise 1.12. *The following problem is known as the Monty Hall problem, after the host of the game show "Let's Make a Deal!". There are three curtains. Behind one curtain is a new car, and behind the other two are goats. The game is played as follows. The contestant chooses the curtain that she thinks the car is behind. Monty then opens one of the other curtains to show a goat. (Monty may have more than one goat to choose from; in this case, assume he chooses which goat to show uniformly at random.) The contestant can then stay with the curtain she originally chose or switch to the other unopened curtain. After that, the location of the car is revealed, and the contestant wins the car or the remaining goat. Should the contestant switch curtains or not, or does it make no different.?*

Solution. The contestant should switch to the other unopened curtain. Of course you can use Bayes' Law to prove this claim. Here we give another explanation from the concept of sample space.

What if the host does NOT know where the car is? Assume that the goats have names as G_A and G_B. Let C denote the car. Consider the Figure 1 to understand what the sample space is.

You pick	Host reveals	The 3rd door contains
G_A	G_B	C
G_B	G_A	C
G_A	C	G_B
G_B	C	G_A
C	G_A	G_B
C	G_B	G_A

Figure 1: The sample space when the host does not know where the car is.

It is obvious that the probability of getting a car when you change your mind is $2/6 = 1/3$. However, if the host knows where the car is, the sample space changes since he or she must reveal the door of goats. Consider the Figure 2 for an illustration.

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You pick	Host reveals	The 3rd door contains
G_A	G_B	C
G_B	G_A	C
G_A	G_B	C
G_B	G_A	C
C	G_A	G_B
C	G_B	G_A

Figure 2: The sample space when the host does knows where the car is.

As Figure 2 shows, the probability of getting a car when you change your mind increases to $4/6 = 2/3$.

What confuse the readers (or audience) is that the description of the Monty Hall problem is not precise and clear. We do not know whether the host is honest or not, and whether he knows where the car is or not. Even though we have the solution obtained by Bayes' theorem, it assumes that *the host knows where the car is* at first. This paradox teaches us that the paradox sometimes comes from misleading or imprecise descriptions. □

Exercise 1.13. *A medical company touts its new test for a certain genetic disorder. The false negative rate is small: if you have the disorder, the probability that the test returns a positive result is 0.999. The false positive rate is also small: if you do not have the disorder, the probability that the test returns a positive result is only 0.005. Assume that 2% of the population has the disorder. If a person chosen uniformly from the population is tested and the result comes back positive, what is the probability that the person has the disorder?*

***Solution*.** Let us define some events first.

D: a person chosen uniformly at random has the disorder;

P: the test result for a person is positive;

N: the test result for a person is negative.

Hence from the description of the problem, we have

$$\begin{aligned}
\mathbf{Pr}[P \mid D] &= 0.999 \\
\mathbf{Pr}[P \mid \bar{D}] &= 0.005 \\
\mathbf{Pr}[D] &= 0.02 \\
\mathbf{Pr}[\bar{D}] &= 0.98
\end{aligned}$$

Thus we have what we want to know is $\mathbf{Pr}[D \mid P]$, which is calculated as follows (by Bayes' Law).

$$\begin{aligned}
\mathbf{Pr}[D \mid P] &= \frac{\mathbf{Pr}[D \cap P]}{\mathbf{Pr}[P]} \\
&= \frac{\mathbf{Pr}[P \mid D] \cdot \mathbf{Pr}[D]}{\mathbf{Pr}[P \mid D] \cdot \mathbf{Pr}[D] + \mathbf{Pr}[P \mid \bar{D}] \cdot \mathbf{Pr}[\bar{D}]} \\
&= \frac{0.999 \cdot 0.02}{0.999 \cdot 0.02 + 0.005 \cdot 0.98} \\
&\approx 0.80305466.
\end{aligned}$$

□

Exercise 1.21. *Give an example of three random events X, Y, Z for which any pair are independent but all three are not mutually independent.*

Solution. Let $a_0a_1a_2$ be a 0-1 sequence of length 3. Let X be an event that $a_0 \neq a_1$, Y be an event that $a_1 \neq a_2$, and Z be an event that $a_0 \neq a_2$. It is clear that $\mathbf{Pr}[X] = \mathbf{Pr}[Y] = \mathbf{Pr}[Z] = 1/2$, and $\mathbf{Pr}[X \cap Y] = \mathbf{Pr}[Y \cap Z] = \mathbf{Pr}[X \cap Z] = 1/4$ (this can be verified by simply checking the sample space of size 2^3 and you will obtain two of them matches the event $X \cap Y$, $Y \cap Z$, or $X \cap Z$). However, $\mathbf{Pr}[X \cap Y \cap Z] = 0$ since if $a_0 = a_1, a_1 = a_2$ (i.e., $X \cap Y$ occurs) then $a_0 = a_2$ (i.e., Z never occurs). We have $\mathbf{Pr}[X \cap Y \cap Z] \neq \mathbf{Pr}[X] \cdot \mathbf{Pr}[Y] \cdot \mathbf{Pr}[Z]$. Therefore, the three events X, Y, Z are what we want. □

Exercise 1.22.

(a) Consider the set $\{1, \ldots, n\}$. We generate a subset X of this set as follows: a fair coin is flipped independently for each element of the set; if the coin lands heads then the element is added to X, and otherwise it is not. Argue that the resulting set X is equally likely to be any one of the 2^n possible subsets.

(b) Suppose that two sets X and Y are chosen independently and uniformly at random from all the 2^n subsets of $\{1, \ldots, n\}$. Determine $\mathbf{Pr}[X \subseteq Y]$ and $\mathbf{Pr}[X \cup Y = \{1, \ldots, n\}]$ (Hint: Use the part (a) of this problem).

Solution.

(a) Let $v_X = (x_1, x_2, \ldots, x_n)$ be a *set vector* showing the result of generating X by the algorithm, where $\mathbf{Pr}[x_i = 0] = \mathbf{Pr}[x_i = 1] = 1/2$. What we have to prove is that every subset $S \subseteq \{1, \ldots, n\}$ has the same probability of being generated. That is, $\mathbf{Pr}[X = S] = 1/2^n$. S can be viewed as an ordered 0-1 sequence $s_1, s_2, \ldots, s_n$, where $s_i = 0$ or 1, and $s_i = 1$ if and only if $i \in S$. S is exactly X if $x_i = s_i$ for each i. Since
$$\mathbf{Pr}[x_i = s_i] = \frac{|\{(0,0),(1,1)\}|}{|\{(0,0),(0,1),(1,0),(1,1)\}|} = \frac{1}{2},$$
Hence we have $\mathbf{Pr}[X = S] = \mathbf{Pr}\left[\cap_{i=1}^{n} x_i = s_i\right] = \prod_{i=1}^{n} \mathbf{Pr}[x_i = s_i] = (1/2)^n$ (since the coin is flipped independently for each element). Therefore, the claim of the problem is proved.

(b) As we have defined in (a), let $v_X = (x_0, \ldots, x_n)$ and $v_Y = (y_0, \ldots, y_n)$ be the corresponding set vector of X and Y respectively. Assume that $i_0, \ldots, i_k \in X$, that is, X contains k elements of $\{1, \ldots, n\}$, so $x_{i_j} = 1$ for $j = 0, \ldots, k$. If $X \subseteq Y$, then y_{i_j} has to be 1 for each $j = 0, \ldots, k$, and y_r can be 0 or 1 for $r \in \{1, \ldots, n\} \setminus \{i_0, \ldots, i_k\}$. Note that k, which stands for the size of X, ranges from 0 to n, so the number of possible pairs of (X, Y) satisfying $X \subseteq Y$ is $\sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 2^{n-k}$. Hence we have

$$\begin{aligned}
\mathbf{Pr}[X \subseteq Y] &= \frac{\sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 2^{n-k}}{2^n \cdot 2^n} \\
&= \frac{\sum_{k=0}^{n} \binom{n}{k} 2^{-k}}{2^n} \\
&= \frac{(1 + 1/2)^n}{2^n} \\
&= \left(\frac{3}{4}\right)^n \\
&= \frac{1}{(4/3)^n}.
\end{aligned}$$

Consider the case that $X \cup Y = \{1, \ldots, n\}$. As the above analysis, we assume that $i_0, i_1, \ldots, i_k \in X$ so $x_{i_j} = 1$ for $j = 0, \ldots, k$. Here y_{i_j} has to be 1 for $j \in \{1, \ldots, n\} \setminus \{i_0, \ldots, i_k\}$, and y_r can be 0 or 1 for $r \in \{i_0, i_1, \ldots, i_k\}$. Hence we have

$$\begin{aligned}
\mathbf{Pr}[X \cup Y = \{1, \ldots, n\}] &= \frac{\sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot 2^k}{2^n \cdot 2^n} \\
&= \frac{\sum_{k=0}^{n} \binom{n}{k} 2^k}{4^n} \\
&= \frac{(2 + 1)^n}{4^n} \\
&= \left(\frac{3}{4}\right)^n \\
&= \frac{1}{(4/3)^n}.
\end{aligned}$$

□

Exercise 1.23. *There may be several different min-cut sets in a graph. Using the analysis of the randomized min-cut algorithm, argue that there can be at most $n(n-1)/2$ distinct min-cut sets.*

Solution. Every possible min-cut can be generated by the randomized min-cut algorithm (as long as the edges in the min-cut set are not contracted). Thus every min-cut set can be generated (with probability at least $2/(n(n-1))$). Note that at the end of the execution of the algorithm there are two vertices left (as well as the multi-edges standing for a cut set). In addition, there are $\binom{n}{2}$ pairs of vertices in an n-vertex graph, so there are at most $\binom{n}{2}$ different result of the algorithm, each of which stands for a candidate of min-cuts. Therefore the number of distinct min-cut sets are at most $n(n-1)/2$. □