Exercises of Chapter 6

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Exercise 6.16. *Use the Lovasz local lemma to show that, if*

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \le 1,$$

then it is possible to color the edges of K_n with two colors so that it has no monochromatic K_k subgraph.

Solution. There are $\binom{n}{k}$ K_k cliques in K_n. We let A_i be a bad event such that the ith clique K_k is monochromatic. Since each clique K_k has $\binom{k}{2}$ edges, $\mathbf{Pr}[A_i] = 2/2^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$. We can construct a dependency graph $G = (V, E)$, where each vertex $v_i \in V$ corresponds to the event A_i. Furthermore, $(v_i, v_j) \notin E$ if and only if A_i and A_j are independent. Note that for a fixed clique, the number of other cliques sharing at least two edges with it is at most $\binom{k}{2}\binom{n-2}{k-2} < \binom{k}{2}\binom{n}{k-2}$, so we know that each vertex in the dependency graph has degree at most $\binom{k}{2}\binom{n}{k-2}$, i.e., $d \le \binom{k}{2}\binom{n}{k-2}$. Let p denote $\mathbf{Pr}[A_i]$. Since $4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \le 1$, we have $4dp \le 1$. Hence by Lovasz local lemma, it is possible that none of the bad events (i.e., A_i's) happens, that is, there exists a monochromatic K_k subgraph in K_n. $\square$

Exercise 6.18. *Let $G = (V, E)$ be an undirected graph and suppose each $v \in V$ is associated with a set $S(v)$ of $8r$ colors, where $r \ge 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most r neighbors u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$ such that, for any edge $(u, v) \in E$, the colors assigned to u and v are different. You may want to let $A_{u,v,c}$ be the event that u and v are both colored with color c and then consider the family of such events.*

Solution. As the hint given in the problem description, we let $A_{u,v,c}$ be the *bad event* that u and v, where u and v are adjacent, are both colored with color c. It is clear that the event happens only when the color c lies in both $S(u)$ and $S(v)$. If $c \notin S(u)$ or $c \notin S(v)$, then $\mathbf{Pr}[A_{u,v,c}] = 0$, so we consider the case that $c \in S(u)$ and $c \in S(v)$. We derive that

$$\mathbf{Pr}[A_{u,v,c}] \le \frac{1}{(8r)^2} = \frac{1}{64r^2}.$$

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We can construct a dependency graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ consists of the events $\{A_{u,v,c} \mid (u,v) \in E\}$. Since for each $v \in V$ and $c \in S(v)$ there are at most r neighbors u of v such that c lies in $S(u)$, we have that $A_{u,v,c}$ has dependency on at most $8r \cdot r + 8r \cdot r = 16r^2$ other events, the degree of $\mathcal{G}$, i.e., d, is at most $16r^2$. Since $4 \cdot \mathbf{Pr}[A_{u,v,c}] \cdot d \leq 4 \cdot (1/64r^2) \cdot 16r^2 \leq 1$, by Lovasz local lemma, we have the desired result. $\square$

Extra problem. *Suppose that we have a set S of numbers $\{x_1, x_2, \ldots, x_n\}$ and we want to select one of them that belongs to the "upper half" (i.e., it is greater than or equal to the median).*

(a) Prove that is is impossible to guarantee that a number belongs to the upper half by making less than $n/2$ comparisons.

(b) Give a Monte-Carlo algorithm to obtain a number that belongs to the upper half with high probability.

Solution.

(a) Let $X = \{x_1, x_2, \ldots, x_n\}$. For any algorithm to select a number that belongs to the "upper half" of X, *comparisons* must be performed, and to ensure a number x_i belongs to the upper half, one must guarantee that x_i is greater than $\lfloor n/2 \rfloor$ numbers in X. However, for any algorithm solving this problem, we can always create a *worst case* input such that the first $\lfloor n/2 \rfloor$ numbers chosen by the algorithm belong to the lower half of X. That is, even though the algorithms choose the largest number of them by applying $\lfloor n/2 \rfloor - 1 < \lfloor n/2 \rfloor \leq n/2$ comparisons, it can not obtain a number belonging to the upper half.

(b) We propose a very simple algorithm as follows. The algorithm *Monte-Carlo upper-*

Monte-Carlo upper-half-choosing (X)
Choose two numbers $x_i, x_j \in X$ uniformly at random.
Return $\max\{x_i, x_j\}$;

half-choosing only costs *one* comparison. The algorithm errors only when two chosen numbers x_i, x_j are both in the lower half of X. Hence the error probability of the algorithm is $\mathbf{Pr}[\{x_i \text{ is in the lower half of } X\} \cap \{x_j \text{ is in the lower half of } X\}] < 1/2 \cdot 1/2 = 1/4$.

$\square$