Randomized Algorithms

Parrondo's Paradox

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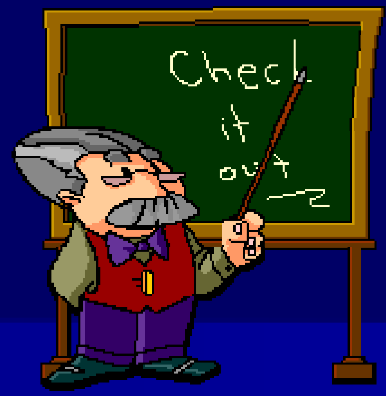
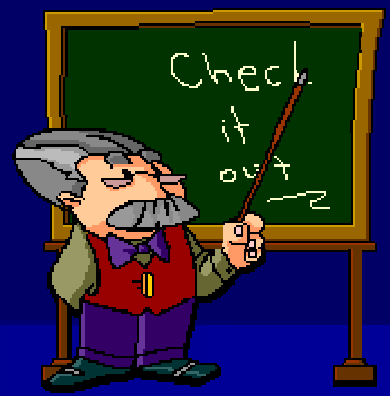
References



- Professor S. C. Tsai's slides.
- *Randomized Algorithms*, Rajeev Motwani and Prabhakar Raghavan.
- *Probability and Computing - Randomized Algorithms and Probabilistic Analysis*, Michael Mitzenmacher and Eli Upfal.

Outline



- Introduction
- Two games
  - Game *A*
  - Game *B*
- Combining Game *A* and *B*
  - Two losing games become a winning game

Introduction

- Parrondo's paradox provides an interesting example of the analysis of Markov chains while also demonstrating a subtlety in dealing with probabilities.

- The paradox appears to contradict the old saying that two wrongs don't make a right.

Introduction (cont'd)

- Because Parrondo's paradox can be analyzed in many different ways, we will go over several approaches to this problem.
- Let us see the first game, i.e., game *A*, as follows.

Game *A*

- Repeatedly flip a biased coin (coin *a*) that comes up head with probability $p_a < ½$ and tails with probability $1 – p_a$.
- We win one dollar if it comes up "heads" and lose one dollar if it comes up "tails".
- Clearly, this is a losing game for us.

Game *B*

- Repeatedly flip coins, but the coin that is flipped depends on the previous outcomes.
- Let w be the number of our wins so far and l the number of our losses.
- Each round we bet one dollar, so $w - l$ represents winnings; if it is negative, we have lost money

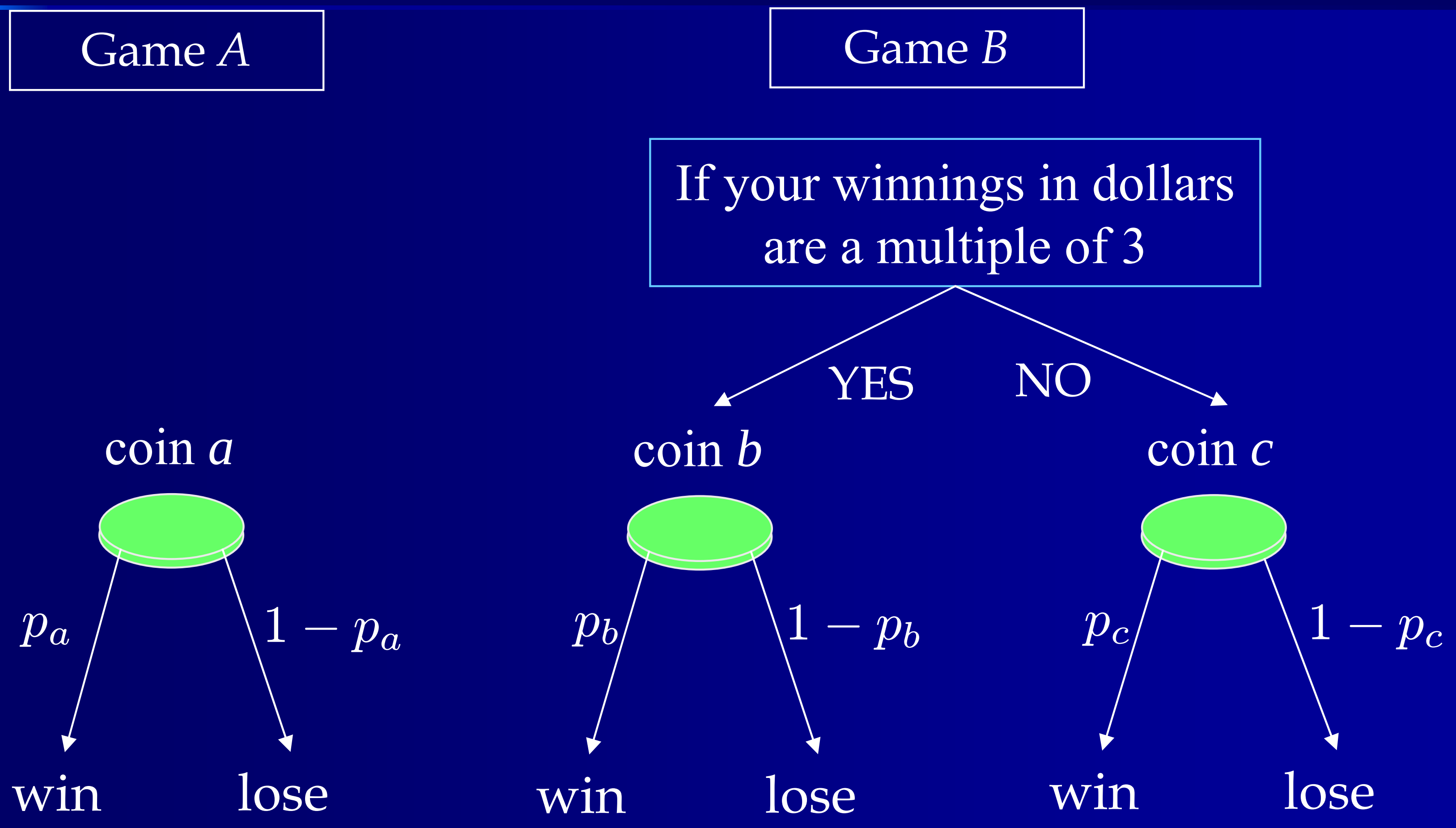
Game *B* (cont'd)

- Game *B* uses two biased coins, say coin *b* and coin *c*.
- If our winnings in dollars are a multiple of 3, then we flip coin *b*, which comes up heads with probability p_b. and tails with probability $1-p_b$.
- Otherwise flip coin *c*, which comes up head with probability p_c and tails with probability $1-p_c$.

Game *B* (cont'd)

- Again, we win one dollar, if it comes up head.
- Let us see the following illustration to make clear of these two games.

An illustration



An example for game *B*

- Suppose $p_b = 0.09$ and $p_c = 0.74$.
- **If** we use coin *b* for 1/3 of the time that the winnings are a multiple of 3 and use coin *c* the other 2/3 of the time.
- The probability of winning is

$$w = \frac{1}{3} \cdot \frac{9}{100} + \frac{2}{3} \cdot \frac{74}{100} = \frac{157}{300} > \frac{1}{2}.$$

An example for game *B* (cont'd)

- But coin *b* may not be used 1/3 of the time!
- Consider the following situation:

An example for game *B* (cont’d)

- Intuitively, when starting with winning 0, use coin *b* and most likely lose, after which use coin *c* and most likely win.

- Thus, we may spend lots of time going back and forth between having lost one dollar and breaking even before either winning one dollar or losing two dollars.

An example for game *B* (cont'd)

- So we may use coin *b* more than 1/3 of the time.
- Suppose we start playing Game *B* when the winning is 0, continuing until either lose three dollars or win three dollars.
- Note that if you are more likely to lose than win in this case, by symmetry you are more likely to lose 3 dollars than win 3 dollars whenever $3 \mid w - l$.

An example for game *B* (cont'd)

- In fact, this specific example for game B is a losing game for us.

- Let consider the following two ways to analyze this phenomenon by Markov chains.
  - Analyze the absorbing states.
  - Use the stationary distribution

Analyzing the absorbing states

- Consider the Markov chain on the state space consisting of the integers $\{-3, -2, -1, 0, 1, 2, 3\}$.
  - The states represent our winnings.
- We show that it is more likely to reach -3 than 3.

Analyzing the absorbing states (cont'd)

- Let z_i be the probability that the game will reach –3 before reaching 3 when starting with winning i.

- We want to calculate z_i, for $i = -3, \ldots, 3$, especially z_0.

- $z_0 > ½$ means it is more likely to lose three dollars before winning three dollars starting from 0.

Analyzing the absorbing states (cont'd)

- Note that $z_{-3} = 1$ and $z_3 = 0$
  - Boundary conditions
- We have the following equations:

$$
\begin{aligned}
z_{-2} &= (1-p_c)z_{-3} + p_c z_{-1}, \\
z_{-1} &= (1-p_c)z_{-2} + p_c z_0, \\
z_0 &= (1-p_b)z_{-1} + p_b z_1, \\
z_1 &= (1-p_c)z_0 + p_c z_2, \\
z_2 &= (1-p_c)z_1 + p_c z_3.
\end{aligned}
$$

Analyzing the absorbing states (cont'd)

- The is a system of five equations with five unknown variables, hence it can be solved easily.
- We have the general solution for z_0 is

$$z_0 = \frac{(1-p_b)(1-p_c)^2}{(1-p_b)(1-p_c)^2 + p_b p_c^2}.$$

- So the solution yields $z_0 = 15379/27700 \approx 0.555$, showing that we are much more to lose than to win playing game *B*.

Using Stationary Distribution

- Consider the Markov chain on the states {0, 1, 2}.
  - Each state keeps track of $(w - l)$ mod 3.

- Let π_i's be the stationary probabilities of this chain.

Using Stationary Distribution (cont'd)

- The probability that we win one dollar in the stationary distribution (which is the limiting probability that we win one dollar if we play long enough) , is

$$
\begin{aligned}
& p_b\pi_0 + p_c\pi_1 + p_c\pi_2 \\
= \quad & p_b\pi_0 + p_c(1 - \pi_0) \\
= \quad & \boxed{p_c - (p_c - p_b)\pi_0.}
\end{aligned}
$$

We wonder whether the value is > ½ or < ½ .

Using Stationary Distribution (cont'd)

- The equations for the stationary distribution are as follows:

$$
\begin{aligned}
\pi_0 + \pi_1 + \pi_2 &= 1 \\
p_b \pi_0 + (1 - p_c)\pi_2 &= \pi_1, \\
p_c \pi_1 + (1 - p_b)\pi_0 &= \pi_2, \\
p_c \pi_2 + (1 - p_c)\pi_1 &= \pi_0.
\end{aligned}
$$

- Since there are four equations and only three unknown variables, this system can be solved easily.

Using Stationary Distribution (cont'd)

- Thus we have

$$\pi_0 = \frac{1 - p_c + p_c^2}{3 - 2p_c - p_b + 2p_bp_c + p_c^2},$$
$$\pi_1 = \frac{p_bp_c - p_c + 1}{3 - 2p_c - p_b + 2p_bp_c + p_c^2},$$
$$\pi_2 = \frac{p_bp_c - p_b + 1}{3 - 2p_c - p_b + 2p_bp_c + p_c^2}.$$

- Pluggin $p_b = 0.09$ and $p_c = 0.74$, we have

$$\pi_0 = 673/1759 \approx 0.3826\ldots$$

Using Stationary Distribution (cont'd)

- Thus

$$p_c - (p_c - p_b)\pi_0 = \frac{86421}{175900} < \frac{1}{2}.$$

Again, we find that game *B* in this case is a losing game in the long run.

Consider what happens when we combine these two games?

Game *C* : Combining game *A* and *B*

- <u>Game *C*</u>: Repeatedly perform the following:

Start by ° ipping a fair coin d.

★ If d comes out head, then proceed as in game A

★ If d comes out tail, then proceed to game B.

It seems that game C is a losing game, right?

Game *C* (cont'd)

- Let us check it with the Markov chain approach, by analyzing the stationary distribution.

- If $3 \mid w - l$, then we win with probability $p_b^* = ½\, p_a + ½\, p_b$.
- Otherwise, the probability that we win is $p_c^* = ½\, p_a + ½\, p_c$.

- Thus we can use p_b^* and p_c^* in place of p_b and p_c.

Game *C* (cont'd)

- By the previous analysis, we can focus on the value

$$p_c^* - (p_c^* - p_b^*)\pi_0$$

- By plugging p_b^* and p_c^* which can be calculated from p_a, p_b and p_c, we have

$$p_c^* - (p_c^* - p_b^*)\pi_0 = \frac{4456523}{8859700} > \frac{1}{2}.$$

So game *C* appears to be a winning game!

However,

- You may be concerned that this seems to violate the law of linearity of expectations
- As the following:

$$\mathbf{E}[X_C] = \mathbf{E}[\frac{1}{2}X_A + \frac{1}{2}X_B] = \frac{1}{2}\mathbf{E}[X_A] + \frac{1}{2}\mathbf{E}[X_B].$$

But $\mathbf{E}[X_A] < 0$, $\mathbf{E}[X_B] < 0$, how can $\mathbf{E}[X_C] > 0$??

Explanation

- The problem is that this equation does not make sense.
- We cannot talk about the expected winnings of a round of games *B* and *C* **without reference to the current winnings**.
- Let *s* represent the current state. We have

$$\mathbf{E}[X_C \mid s] = \mathbf{E}[\frac{1}{2}(X_A + X_B) \mid s] = \frac{1}{2}\mathbf{E}[X_A \mid s] + \frac{1}{2}\mathbf{E}[X_B \mid s].$$

Explanation (cont'd)

- Linearity holds for any given step; but we must condition on the current state.

Conclusion

- Combining the games we've changed how often the chain spends in each state, allowing two losing games to become a winning game!

- It is quite interesting.

Thank you.